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Noncompact $SL(2,\mathbb{R})$ spin chain

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Abstract:

We consider the integrable spin chain model – the noncompact $SL(2,\mathbb{R})$ spin magnet. The spin operators are realized as the generators of the unitary principal series representation of the $SL(2,\mathbb{R})$ group. In an explicit form, we construct $\mathcal{R}-$matrix, the Baxter $\mathbb{Q}-$operator and the transition kernel to the representation of the Separated Variables (SoV). The expressions for the energy and quasimomentum of the eigenstates in terms of the Baxter $\mathbb{Q}-$operator are derived. The analytic properties of the eigenvalues of the Baxter operator as a function of the spectral parameter are established. Applying the diagrammatic approach, we calculate Sklyanin's integration measure in the separated variables and obtain the solution to the spectral problem for the model in terms of the eigenvalues of the $\mathbb{Q}-$operator. We show that the transition kernel to the SoV representation is factorized into a product of certain operators each depending on a single separated variable.

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Contents

1. Introduction

Solutions of many integrable models can be obtained within the Algebraic Bethe Ansatz (ABA) method [1]. However, in some cases, this method is not applicable. This happens, in particular, when the Hilbert space of the model does not possess the lowest weight vector. In this case the solution of the model should rely on the methods of the Baxter $\mathbb{Q}-$operator [2] and the Separated Variables (SoV) [3, 4], which present an alternative to the ABA approach. The advantage of these two methods is that their applicability is not subjected to the restrictions of the ABA method, while the disadvantage is that the Baxter $\mathbb{Q}-$operator and representation of the Separated Variables are known only for a limited class of models. The most well known model of this class is the quantum Toda chain. The method of SoV was developed by Sklyanin [3] and worked out for the Toda chain by Kharchev and Lebedev [5]. The Baxter $\mathbb{Q}-$operator for the Toda chain was constructed by Pasquier and Gaudin [6].

Recently the Baxter $\mathbb{Q}-$operator and representation of the Separated Variables have been constructed for a number of models [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Among these models there are so-called noncompact spin magnets. They are the cousins of the conventional spin magnets like the famous XXX Heisenberg spin magnet. The difference between these models is that the spin generators act in different Hilbert spaces. In the case of the compact magnets it is the finite dimensional space of certain representations of the $SU(2)$ group. The Hilbert space of the noncompact magnets is taken to be the vector space of the unitary representations of the noncompact group $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$, which are necessarily infinite-dimensional.

Compact $SU(2)$ spin magnets are of special interest in statistical physics (see e.g. Ref. [16]). The recent interest in the study of the noncompact magnets originates from high energy physics (see review Refs. [17, 18]). The studies of the scale dependence of scattering amplitudes in Quantum Chromodynamics revealed the surprising fact that the equations governing the scale

dependence of many physically important amplitudes are integrable. Namely, it has been shown that the Hamiltonian governing the Regge-behaviour of the hadronic amplitudes is equivalent to the Hamiltonian of the noncompact $SL(2,\mathbb{C})$ spin magnet [19, 20]. Further, it was observed in [21] that the scale dependence of certain partonic distributions in QCD is governed by Hamiltonians which are equivalent to those of the spin chains, both closed and open, with $SL(2,\mathbb{R})$ symmetry group. The Hilbert space of the latter is given by the tensor product of the vector spaces of the discrete series representations of the $SL(2,\mathbb{R})$ group and possesses the lowest weight vector. Therefore the solution of these models can be obtained by the ABA method (see Refs. [22, 23, 24] for the analysis of these particular models). Contrary, the solution of the $SL(2,\mathbb{C})$ spin magnets relies entirely on the method of the Baxter $\mathbb{Q}-$operator and Separation of Variables [11].

A general method of constructing the Baxter $\mathbb{Q}-$operator is not developed yet and the latter is known for a limited class of models only. For the noncompact $SL(2,\mathbb{R})$ spin chain the Baxter $\mathbb{Q}-$operator was first constructed in Ref. [7]. The approach used in [7] is based on the Pasquier and Gaudin method [6] and suggests an effective way to resolve the defining equations and obtain the integral kernel of the $\mathbb{Q}-$operator in an explicit form. Further, in the case of the noncompact spin magnets it appears to be quite helpful to interpret the integral kernels of the operators in question ($\mathbb{Q}-$operator, transfer matrix, etc.) as Feynman diagrams of a certain type [11, 12]. Then the analysis of the properties of the models is drastically simplified and in many cases can be fulfilled diagrammatically. Moreover, following this approach one can construct not only the Baxter operator that allows to determine the energies of the eigenstates, but also the transition operator to the SoV representation [11, 12, 15] that gives the explicit representation for the eigenfunctions of the model. Surprisingly, such a reformulation appears especially effective for the models where the ABA method does not work, e.g. $SL(2,\mathbb{C})$ spin magnet [11].

In the present paper we consider the noncompact spin magnet with the Hilbert space given by the tensor product of the vector spaces of the unitary principal series representation of $SL(2,\mathbb{R})$ group. Such models have not been considered so far and differ drastically in their properties from the $SL(2,\mathbb{R})$ magnet considered in the literature. A particular choice of the Hilbert space gives this model some similarity with the $SL(2,\mathbb{C})$ spin magnets. This similarity also appears in the method of the analysis used. The solution in both cases relies on the Baxter $\mathbb{Q}-$operator and SoV methods. However, this model possesses some specific properties which make it different from both $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$ (discrete series) spin magnets. The two-particle Hamiltonian has both the discrete and continuous spectrum, the energy of the corresponding eigenstate is not completely fixed by its two-particle spin and deviates from the usual $\psi-$function form. We found that such fundamental object as $\mathcal{R}-$operator, Baxter $\mathbb{Q}-$operator are doubled, i.e. there are two independent solutions of the Yang-Baxter relation for $\mathcal{R}-$operator and two Baxter $\mathbb{Q}-$operators. The latter satisfy the Baxter equation which entangles them. Although it is possible to introduce a linear combination of the Baxter operators such that each satisfies the separate equation, they remain entangled through the quantization conditions.

The paper is organized as follows: In the Sect. 2. we remind some facts about $SL(2,\mathbb{R})$ group and introduce the notations. In the Sect. 3. we construct the $SL(2,\mathbb{R})$ invariant solution of the Yang-Baxter relation and discuss its properties. The transfer matrices and Hamiltonian are built in the Sect. 3.2.. In the Sect. 4. we construct the Baxter $\mathbb{Q}-$operator and study its properties. The representation of the Separated Variables is constructed in the Sect.5.. The eigenvalues of the Baxter $\mathbb{Q}-$operator can be found in analytic form for the two-site chain, which is considered in the Sect. 6.. The concluding remarks are given in the Sect. 7. and the Appendix contains some technical details.

2. Preliminaries

The aim of this section is to remind some basic facts about the representations of the group $SL(2,\mathbb{R})$ and introduce the necessary notations. It is well known that all unitary representations of the group of the real unimodular matrices $SL(2,\mathbb{R})$ can be organized into three series, the discrete ones, the principal and supplementary continuous series [25]. The latter will not appear in our analysis and therefore is not considered here.

The unitary representation of the principal continuous series is determined by two numbers, a real ρ and a discrete ϵ, which takes only two values 0 and 1/2. It is convenient to denote the pair of numbers (α, ϵ), where α is complex and ϵ is 0 or 1/2, by the bold letter $\boldsymbol{\alpha}$. We define the sum of two numbers $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ as follows

$$\boldsymbol{\alpha_1} + \boldsymbol{\alpha_2} = \boldsymbol{\alpha_3} = (\alpha_1 + \alpha_2, \epsilon_3)\,. \tag{2.1}$$

where $\epsilon_3 = 0$, if $\epsilon_1 + \epsilon_2$ is integer, and $\epsilon_3 = 1/2$ otherwise. In what follows writing the sum $(\epsilon_1 + \epsilon_2)$ we shall always imply such addition rule. Next, the zero element is defined as $\mathbf{0} = (0,0)$. By $-\boldsymbol{\alpha}$ we shall denote the element inverse to $\boldsymbol{\alpha}$, $\boldsymbol{\alpha} + (-\boldsymbol{\alpha}) = \mathbf{0}$, $-\boldsymbol{\alpha} = (-\alpha, \epsilon)$. If the first element, α, of the number $\boldsymbol{\alpha}$ is real, we shall call the number $\boldsymbol{\alpha}$ real as well. The usefulness of such notations will be clear later.

Thus the unitary representation of the principal continuous series is labelled by the real $\boldsymbol{\rho}$. It can be realized by the unitary operators $T^{\boldsymbol{\rho}}(g)$ acting on the Hilbert space $L^2(\mathbb{R})$ [25]

$$\left[T^{\boldsymbol{\rho}}(g)\Psi\right](x) \;=\; \frac{\sigma_\epsilon(cx+d)}{|cx+d|^{1+2i\rho}}\,\Psi\left(\frac{ax+b}{cx+d}\right)\,, \tag{2.2}$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the sign factor is $\sigma_\epsilon(x) = [\mathrm{sign}(x)]^{2\epsilon}$. All representations $T^{\boldsymbol{\rho}}$ (except of the representation $T^{(0,1/2)}$) are irreducible. The two representations $T^{\boldsymbol{\rho}}$ and $T^{\boldsymbol{\rho}'}$ are not equivalent provided that $\boldsymbol{\rho} \neq -\boldsymbol{\rho}'$. The unitary operator $\mathcal{M}_{\boldsymbol{\rho}}$ which intertwines the representations $T^{\boldsymbol{\rho}}$ and $T^{-\boldsymbol{\rho}}$ ($T^{-\boldsymbol{\rho}}\mathcal{M}_{\boldsymbol{\rho}} = \mathcal{M}_{\boldsymbol{\rho}} T^{\boldsymbol{\rho}}$) is defined uniquely up to a phase factor [25]

$$\left[\mathcal{M}_{\boldsymbol{\rho}}\Psi\right](x) \;=\; \left(\sqrt{\pi}\,A(\boldsymbol{\gamma})\right)^{-1}\int_{-\infty}^{\infty} dy\,\Psi(y)\,\frac{\sigma_\epsilon(x-y)}{|x-y|^{2(1-s)}}\,, \tag{2.3}$$

where the parameter s (conformal spin) is determined as $s \equiv 1/2 + i\rho$ and $\boldsymbol{\gamma} = (2-2s, \epsilon)$. The function $A(\boldsymbol{\alpha})$ is given by the following expression

$$A(\boldsymbol{\alpha}) \;=\; e^{-i\pi\epsilon}\left(\frac{\alpha}{2}\right)^{2\epsilon} a(\epsilon + \alpha/2)\,, \tag{2.4}$$

where $a(x) = \Gamma(1/2 - x)/\Gamma(x)$. This function, which appears naturally in the course of the calculations, satisfies the following conditions

$$A(\boldsymbol{\alpha})\,A(\mathbf{1}-\boldsymbol{\alpha}) \;=\; (-1)^{2\epsilon}\,, \qquad A(\boldsymbol{\alpha})^* \;=\; (-1)^{2\epsilon}\,A(\boldsymbol{\alpha}^*)\,, \tag{2.5}$$

where $\boldsymbol{\alpha}^* = (\alpha^*, \epsilon)$. The normalization factor chosen in (2.3) ensures that the intertwining operator satisfies the following equation $\mathcal{M}_\rho^\dagger = \mathcal{M}_{-\rho}$ [1].

[1] Henceforth we shall not display explicitly the label $\boldsymbol{\rho}$ of the operator $\mathcal{M}$ since it completely fixed by the transformation properties of the function the operator $\mathcal{M}$ acts on.

The three generators of the $s\ell(2)$ algebra corresponding to the realization of the representation $T^{\boldsymbol{\rho}}$ (2.2) have the form

$$S_- = -\partial_x, \quad S_+ = x^2\partial_x + 2sx, \quad S_0 = x\partial_x + s\,. \tag{2.6}$$

These operators are antihermitean and obey the standard $s\ell(2)$ commutation relations

$$[S_0, S_\pm] = \pm S_\pm, \qquad [S_+, S_-] = 2S_0\,. \tag{2.7}$$

The tensor product of two representations of the principal unitary series can be decomposed into the direct integral of the representations of the same type and the direct sum of the representations of the discrete series, $\mathcal{D}_h^\pm$, [26]

$$T^{\boldsymbol{\rho}_1}\otimes T^{\boldsymbol{\rho}_2} = 2\int_0^\infty d\rho\, T^{\boldsymbol{\rho}} + \sum_{h=1+(\epsilon_1+\epsilon_2)/2}^{\infty} (\mathcal{D}_h^+ \oplus \mathcal{D}_h^-)\,. \tag{2.8}$$

The representation of the continuous series, $T^{\boldsymbol{\rho}}$, enters into the direct integral with the multiplicity two. The operators separating the irreducible components and the other necessary details can be found in the Appendix A.

We conclude this section with the following remark. Let us divide all tensor products $T^{\boldsymbol{\rho_1}}\otimes T^{\boldsymbol{\rho_2}}$ in two subsets depending on the value, 0 or 1/2, of the sum $\epsilon_1+\epsilon_2$. Then it is easy to see that all representations inside each group are unitary equivalent to each other. In particular, the representation of the first group, $\epsilon_1+\epsilon_2=0$, are equivalent to $T^{(0,0)}\otimes T^{(0,0)}$, and those in the second group, $\epsilon_1+\epsilon_2=1/2$, are equivalent to $T^{(0,0)}\otimes T^{(0,1/2)}$ [2]. To show this let us consider the operators $V(\boldsymbol{\alpha})$,

$$[V(\boldsymbol{\alpha})\Psi](x_1,x_2) \;=\; \frac{\sigma_\epsilon(x_1-x_2)}{|x_1-x_2|^{2i\alpha}}\,\Psi(x_1,x_2)\,, \tag{2.9}$$

and $U(\boldsymbol{\alpha})$,

$$U(\boldsymbol{\alpha}) \;=\; (\mathcal{M}\otimes\mathbb{I})\,V(\boldsymbol{\alpha})\,(\mathcal{M}\otimes\mathbb{I})\,, \tag{2.10}$$

where $\boldsymbol{\alpha}=(\alpha,\epsilon)$ and the operator $\mathcal{M}$ defined in (2.3) intertwines the representations $T^{\boldsymbol{\rho_1}}$ and $T^{-\boldsymbol{\rho_1}}$. It is obvious that for real $\boldsymbol{\alpha}$, the operators $V(\boldsymbol{\alpha})$ and $U(\boldsymbol{\alpha})$ are unitary and that they intertwine the representation $T^{\boldsymbol{\rho}_1}\otimes T^{\boldsymbol{\rho}_2}$ with $T^{\boldsymbol{\rho}_1+\boldsymbol{\alpha}}\otimes T^{\boldsymbol{\rho}_2+\boldsymbol{\alpha}}$ and with $T^{\boldsymbol{\rho}_1-\boldsymbol{\alpha}}\otimes T^{\boldsymbol{\rho}_2+\boldsymbol{\alpha}}$, respectively. Therefore the combination, $U(\boldsymbol{\alpha})\,V(\boldsymbol{\beta})$, with suitably chosen $\boldsymbol{\alpha},\boldsymbol{\beta}$ intertwines any two representations inside each group. We notice also that for a real $\boldsymbol{\alpha}$,

$$V(\boldsymbol{\alpha})^\dagger = V(-\boldsymbol{\alpha}), \qquad U(\boldsymbol{\alpha})^\dagger = U(-\boldsymbol{\alpha})\,. \tag{2.11}$$

3. $\mathcal{R}-$operator

In this section we shall construct the solution of the Yang-Baxter relation

$$\mathcal{R}_{12}(\boldsymbol{u})\,\mathcal{R}_{13}(\boldsymbol{u}+\boldsymbol{v})\,\mathcal{R}_{23}(\boldsymbol{v}) \;=\; \mathcal{R}_{23}(\boldsymbol{v})\,\mathcal{R}_{13}(\boldsymbol{u}+\boldsymbol{v})\,\mathcal{R}_{12}(\boldsymbol{u})\,. \tag{3.1}$$

The operator $\mathcal{R}_{ik}$ in the above equation acts on the tensor product of the spaces $V_i\otimes V_k$ and depends on the spectral parameter $\boldsymbol{u}=(u,\epsilon)$, i.e. $\mathcal{R}(\boldsymbol{u})=\mathcal{R}(u,\epsilon)$. Each space V_i is equivalent

[2] The representation $T^{(0,1/2)}$ is reducible and equivalent to the $\mathcal{D}^+_{1/2}\oplus\mathcal{D}^-_{1/2}$ [25].

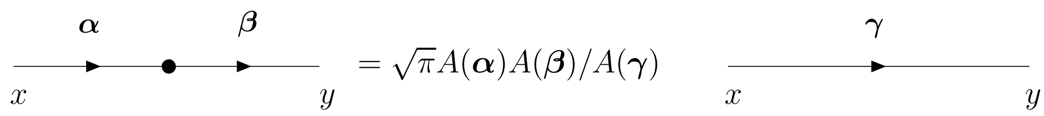

Figure 1: The diagrammatic representation of the chain relation, Eq. (3.9). The black dot denotes the integration vertex and $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta} - \mathbf{1}$.

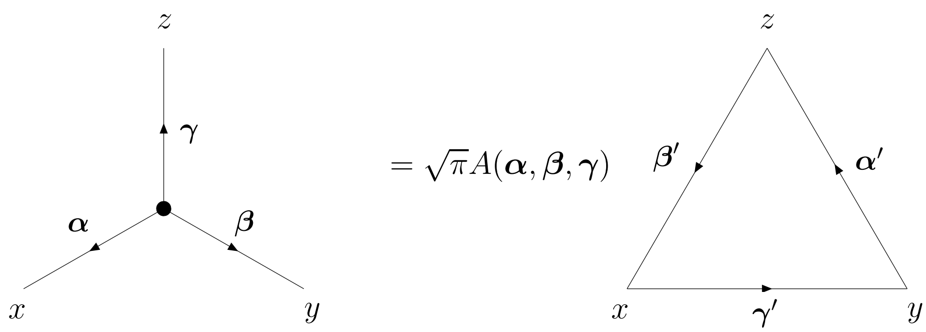

Figure 2: The diagrammatic representation of the star-triangle relation (Eq. (3.10)). The primed indices are defined as $\boldsymbol{x}' = 1 - \boldsymbol{x}$.

to the Hilbert space $L^2(\mathbb{R})$ and carries the representation of the principal continuous series, $T^{\boldsymbol{\rho}_i}$, of the $SL(2,\mathbb{R})$ group. Let us notice that Eq. (3.1) contains two equations involving operators $\widetilde{R}(u) = \mathcal{R}(u,0)$ and $\widehat{R}(u) = \mathcal{R}(u,1/2)$. Indeed, having put $\boldsymbol{u} = (u,0)$ and $\boldsymbol{v} = (v,0)$ in Eq. (3.1) and taking into account that $\boldsymbol{u} + \boldsymbol{v} = (u+v,0)$, one gets the Yang-Baxter equation for the operators $\widetilde{R}$. At the same time, choosing $\boldsymbol{u} = (u,1/2)$ and $\boldsymbol{v} = (v,1/2)$ one finds that the argument of the operator $\mathcal{R}_{13}$ remains the same, $\boldsymbol{u} + \boldsymbol{v} = (u+v,0)$, and therefore, in this case, Eq. (3.1) involves two $\widehat{R}$ operators and one $\widetilde{R}(u+v)$. At a given choice of the spectral parameter these two equations are contained in one equation (3.1). It will be seen later that such, slightly unusual, choice of the spectral parameter is quite natural for this model.

We shall look for the solution of (3.1) in the form of the integral operator

$$\left[\mathcal{R}_{12}(\boldsymbol{u})\Psi\right](x_1,x_2) = \int dy_1\, dy_2\, R_{\boldsymbol{u}}(x_1,x_2|y_1,y_2)\, \Psi(y_1,y_2)\,, \tag{3.2}$$

and impose the additional restrictions of the $SL(2,\mathbb{R})$ invariance. The relation (3.1) leads to certain equations on the integral kernel. We shall propose an ansatz for the kernel $R_{\boldsymbol{u}}(x_1,x_2|y_1,y_2)$, which is motivated by the form of the kernel of the $\mathcal{R}$ operator for the $SL(2,\mathbb{C})$ magnet [11], and show that it satisfies Eq. (3.1). To write down the kernel in a compact form let us define the function (propagator) $D_{\boldsymbol{\alpha}}(x)$,

$$D_{\boldsymbol{\alpha}}(x) = \frac{\sigma_\epsilon(x)}{|x|^\alpha}\,, \tag{3.3}$$

which depends on real x and has the index $\boldsymbol{\alpha} = (\alpha,\epsilon)$. The kernel of the $\mathcal{R}-$matrix is given by the product of four propagators

$$R_{\boldsymbol{u}}(x_1,x_2|y_1,y_2) = r_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(\boldsymbol{u})\, D_{\boldsymbol{\alpha}_4}(y_2-x_1)\, D_{\boldsymbol{\alpha}_3}(x_1-x_2)\, D_{\boldsymbol{\alpha}_2}(x_2-y_1)\, D_{\boldsymbol{\alpha}_1}(y_1-y_2)\,, \tag{3.4}$$

where $r_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(\boldsymbol{u})$ is the normalization coefficient to be defined later and $\boldsymbol{u} = (u,\epsilon)$ is the spectral parameter. The requirement of the $SL(2,\mathbb{R})$ invariance of the kernel imposes the following

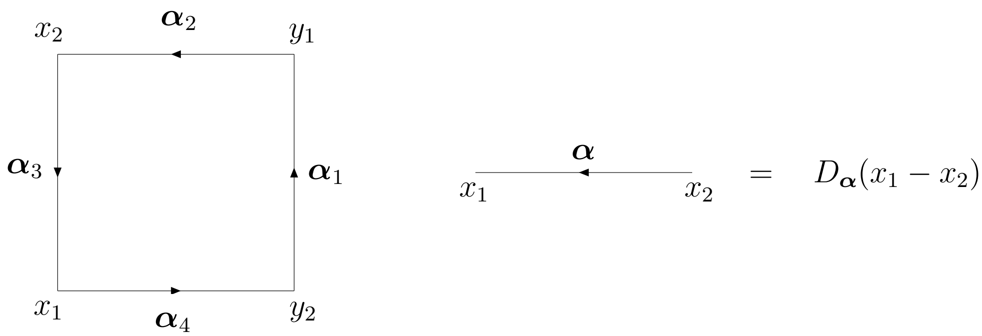

Figure 3: Diagrammatical representation of the kernel of the $\mathcal{R}$-operator, Eq. (3.4).

constraints on the indices $\boldsymbol{\alpha}_i$

$$\begin{aligned} \boldsymbol{\alpha_1} + \boldsymbol{\alpha_2} &= (2-2s_1, \epsilon_1)\,, & \boldsymbol{\alpha_3} + \boldsymbol{\alpha_4} &= (2s_1, \epsilon_1)\,, \\ \boldsymbol{\alpha_1} + \boldsymbol{\alpha_4} &= (2-2s_2, \epsilon_2)\,, & \boldsymbol{\alpha_2} + \boldsymbol{\alpha_3} &= (2s_2, \epsilon_2)\,. \end{aligned} \tag{3.5}$$

Therefore it is sufficient to fix only one index to restore all others. Again, using similarity with the $SL(2,\mathbb{C})$ magnet (see Ref. [11]) we put

$$\boldsymbol{\alpha}_4 = (1 + i\rho_1 - i\rho_2 - iu, \epsilon) \tag{3.6}$$

that gives rises to the following values of the other indices

$$\begin{aligned} \boldsymbol{\alpha}_1 &= (-i\rho_1 - i\rho_2 + iu, \epsilon_2 + \epsilon)\,, \\ \boldsymbol{\alpha}_2 &= (1 - i\rho_1 + i\rho_2 - iu, \epsilon_1 + \epsilon_2 + \epsilon)\,, \\ \boldsymbol{\alpha}_3 &= (i\rho_1 + i\rho_2 + iu, \epsilon_1 + \epsilon)\,. \end{aligned} \tag{3.7}$$

The proof of the Yang-Baxter relation is based on the integral identity for propagators – the so-called star-triangle relation. It is well known in the perturbative Quantum Field Theory and is widely used for the evaluation of the multi-loop Feynman diagrams [27]. As will be seen later, it is quite natural to represent the integral kernels of the operators under consideration in the form of Feynman diagrams. It allows one to carry out an analysis of the properties of the model diagrammatically, which in many cases results in considerable simplifications. Therefore, below we collect some elements of the "diagram technique" we use in our analysis,

- Fourier transform
$$\int dx\, e^{ipx}\, \mathcal{D}_{\boldsymbol{\alpha}}(x) \;=\; \sqrt{\pi}\, 2^{1-\alpha}\, A(\boldsymbol{\alpha})\, \mathcal{D}_{\mathbf{1}-\boldsymbol{\alpha}}(p), \tag{3.8}$$
where $\mathbf{1} = (1,0)$ and the function $A(\boldsymbol{\alpha})$ ($\boldsymbol{\alpha} = (\alpha, \epsilon)$) is defined in Eq. (2.4).

- Chain relation
$$\int dy\, \mathcal{D}_{\boldsymbol{\alpha}}(x-y)\mathcal{D}_{\boldsymbol{\beta}}(y-z) = \sqrt{\pi}\frac{A(\boldsymbol{\alpha})A(\boldsymbol{\beta})}{A(\boldsymbol{\gamma})}\, \mathcal{D}_{\boldsymbol{\gamma}}(x-z), \tag{3.9}$$
where $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta} - \mathbf{1}$.

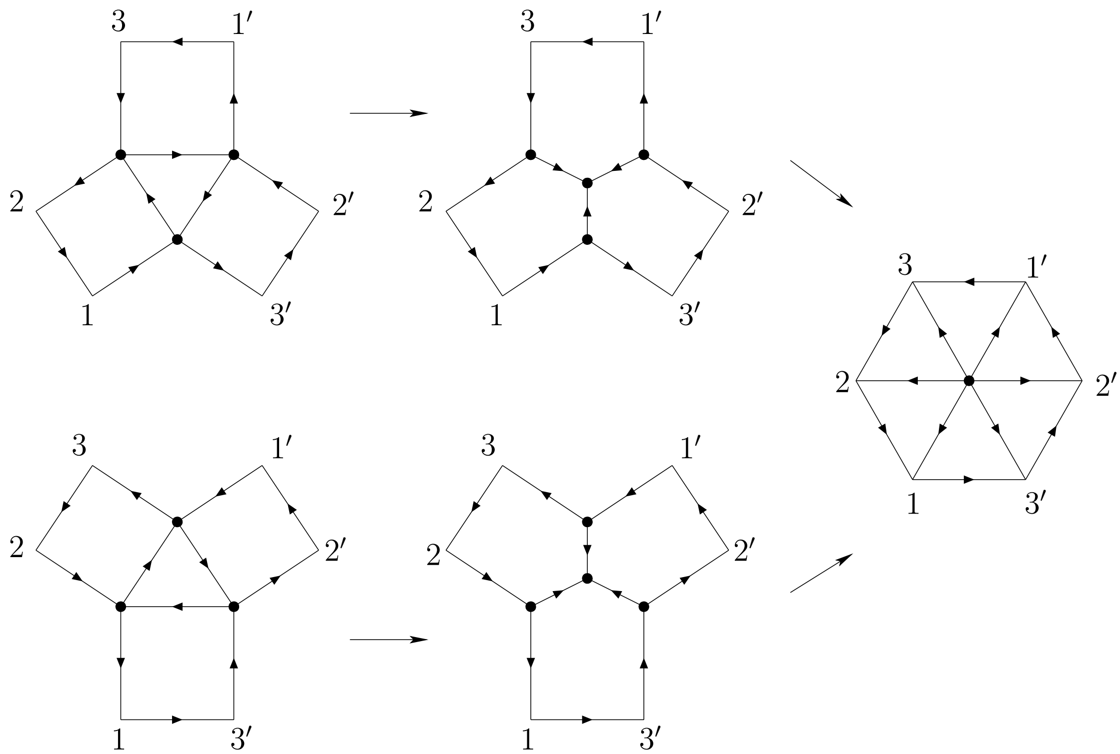

Figure 4: The diagrammatic proof of the Yang-Baxter relation, Eq. (3.1).

- Star-triangle relation

$$\int dw\, \mathcal{D}_{\boldsymbol{\alpha}}(x-w)\mathcal{D}_{\boldsymbol{\beta}}(y-w)\mathcal{D}_{\boldsymbol{\gamma}}(z-w) = \tag{3.10}$$
$$= \sqrt{\pi}\, A(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\, \mathcal{D}_{\mathbf{1}-\boldsymbol{\alpha}}(z-y)\mathcal{D}_{\mathbf{1}-\boldsymbol{\beta}}(x-z)\mathcal{D}_{\mathbf{1}-\boldsymbol{\gamma}}(y-x).$$

  Here $A(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) \equiv A(\boldsymbol{\alpha})A(\boldsymbol{\beta})A(\boldsymbol{\gamma})$ and indices $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ satisfy the *uniqueness* condition $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma} = (2,0)$.

The diagrammatic representation of the above identities is given in Figs. 1 and 2. There the arrow directed from point y to x and labelled by the index $\boldsymbol{\alpha}$ denotes the propagator $\mathcal{D}_{\boldsymbol{\alpha}}(x-y)$ and the black dot is used for the integration vertex.

At last, we give the following representation for the delta function

$$\delta(x) \;=\; \lim_{\alpha\to 0} \frac{a(\alpha/2)}{\sqrt{\pi}}\,\frac{1}{|x|^{1-\alpha}}. \tag{3.11}$$

which can be obtained from Eq. (3.8).

The proof of the Yang-Baxter relation for the $\mathcal{R}-$operator defined in Eq. (3.4) can be carried out diagrammatically. To this end let us notice that graphically the $\mathcal{R}$ operator is represented by the box diagram (see Fig. 3). Then the lhs and rhs of the Yang-Baxter relation (3.1) are represented by the diagrams shown in the the Fig. 4, left up and lower diagrams, respectively.

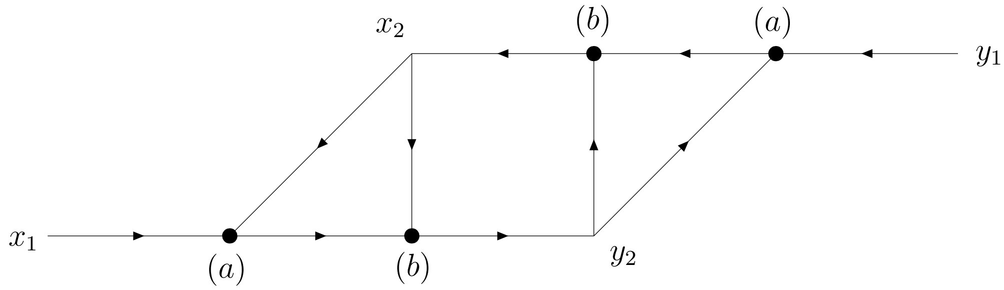

Figure 5: The diagrammatic representation of the rhs of Eqs. (3.14).

To prove the Yang-Baxter equation one has to show that these diagrams are equal. The proof repeats the one given in Ref. [11] and is based on the use of the star-triangle relation (3.10). Indeed, an examination of the indices of the central triangles in those diagrams shows that they satisfy the *uniqueness* condition and, therefore, can be replaced by the star diagrams (3.10) (see second up and lower diagrams in Fig. 4). The new triple vertices in each diagram appear to be unique as well. In the next step one replaces the corresponding star subdiagrams by triangles. The resulting diagram, in both cases, is the hexagon diagram. Restoring all factors which arise in the course of above transformations, see Eq. (3.10), one finds that the final diagrams coincide.

One can also show that $\mathcal{R}-$operator satisfies the Yang-Baxter relation involving two Lax operators

$$L_1(u)\, L_2(v+u)\, \mathcal{R}_{12}(\boldsymbol{u}) \;=\; \mathcal{R}_{12}(\boldsymbol{u})\, L_2(v)\, L_1(u)\,, \qquad (3.12)$$

where, as usual, $L_1(u) = L(u)\otimes\mathbb{I}$, $L_2(u) = \mathbb{I}\otimes L(u)$. The Lax operator is defined in the standard way

$$L(u) \;=\; u + i\begin{pmatrix} S_0 & S_- \\ S_+ & -S_0 \end{pmatrix} \qquad (3.13)$$

and is used for the construction of the auxiliary transfer matrix.

Let us note that the above Yang-Baxter relation holds for the both values of the ϵ, which enters the spectral parameter of the $\mathcal{R}-$operator, i.e. Eq. (3.12) holds both for $\mathcal{R}_{12}(u,0)$ and $\mathcal{R}_{12}(u,1/2)$.

3.1. Properties of $\mathcal{R}-$operator

Since the tensor products $T^{\boldsymbol{\rho}_1}\otimes T^{\boldsymbol{\rho}_2}$ fall into two classes of equivalence, it is natural to expect that the $\mathcal{R}$ operators should be unitary equivalent as well. Indeed, it follows directly from the definition of the $\mathcal{R}$ operator (3.4) that $\mathcal{R}_{\boldsymbol{\rho}_1+\boldsymbol{\alpha},\boldsymbol{\rho}_2+\boldsymbol{\alpha}} \sim V(\boldsymbol{\alpha})\,\mathcal{R}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}\,V(-\boldsymbol{\alpha})$. In the general case the following relation holds:

$$\mathcal{R}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(\boldsymbol{u}) = U(\boldsymbol{\alpha})V(\boldsymbol{\beta})\,\mathcal{R}_{(0,0),(0,\epsilon_1+\epsilon_2)}(u,0)\,V(-\boldsymbol{\beta})\,U(-\boldsymbol{\alpha})\,, \qquad (3.14)$$

where $\boldsymbol{\alpha} = ((\rho_2-\rho_1)/2,\epsilon)$ and $\boldsymbol{\beta} = ((\rho_2+\rho_1)/2,\epsilon_1+\epsilon)$. Here some comments are in order. The diagram corresponding to the integral kernel of the rhs of Eq. (3.14) is drawn in Fig. 5. The central box corresponds to the kernel of the operator $V(\boldsymbol{\beta})\mathcal{R}_{(0,0),(0,\epsilon_1+\epsilon_2)}(u,0)V(-\boldsymbol{\beta})$, and the star subdiagrams from the left and right sides represent the kernel of the U operators. The straightforward application of the star-triangle relation (3.10) in a way indicated in Fig. 5 – one applies the star-triangle relation to the vertices (a) and then to the vertices (b) – turns the

diagram shown in Fig. 5 into the box diagram with indices given by (3.6), (3.7). Therefore, up to some numerical factor, the resulting diagram coincides with those for the $\mathcal{R}-$operator.

Let us remind that the normalization factor, $r_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(\boldsymbol{u})$, in Eq. (3.4) is so far arbitrary. Since an arbitrary $\mathcal{R}-$operator is related by Eq. (3.14) to the operator $\mathcal{R}_{(0,0),(0,0)}(u,0)$ or $\mathcal{R}_{(0,0),(0,1/2)}(u,0)$, we fix the normalization of the latter, and for all other operators take Eq. (3.14) for a definition. Namely, we fix the normalization factor in Eq. (3.4) for the two selected operators as follows

$$r_{(0,0),(0,\epsilon_2)}(u,0) \;=\; 2^{-2iu}\frac{e^{i\pi\epsilon_2}}{\pi}\,A(iu,0)\,A(iu,\epsilon_2)\,, \tag{3.15}$$

where the function $A(\boldsymbol{\alpha})$ is introduced in (2.4). Then one finds that the definition of the $\mathcal{R}-$operator via Eq. (3.14) is equivalent to the following choice of the normalization factor in Eq. (3.4)

$$r_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(\boldsymbol{u}) \;=\; \frac{1}{\pi}\,e^{i\pi(\epsilon_1+\epsilon_2)}\,2^{-2iu}\,A(i\rho_2-i\rho_1+iu,\epsilon)\,A(i\rho_1-i\rho_2+iu,\epsilon+\epsilon_1+\epsilon_2)\,. \tag{3.16}$$

We remind that the sum $(\epsilon_1+\epsilon_2)$ is defined by Eq. (2.1). Such a choice of normalization ensures that the $\mathcal{R}-$operator is unitary for the real $\boldsymbol{u}$,

$$\mathcal{R}_{12}(\boldsymbol{u})\mathcal{R}^{\dagger}_{12}(\boldsymbol{u}) = \mathbb{I}\,. \tag{3.17}$$

Using Eq. (3.11) it can be easily checked that for $\boldsymbol{\rho}_1=\boldsymbol{\rho}_2$

$$\mathcal{R}_{12}(\mathbf{0}) \;=\; (-1)^{2\epsilon_1}\,P_{12}\,, \tag{3.18}$$

where P_{12} is the permutation operator, $[P_{12}\Psi](x_1,x_2)=\Psi(x_2,x_1)$.

Moreover, taking into account that $V(-\boldsymbol{\beta})U(-\boldsymbol{\alpha}) = (U(\boldsymbol{\alpha})V(\boldsymbol{\beta}))^{\dagger}$, (see Eq. (2.11)), one concludes that Eq. (3.14) establishes the unitary equivalence between the arbitrary $\mathcal{R}-$operator and the operator $\mathcal{R}_{(0,0),(0,0)}(\boldsymbol{u})$ (or $\mathcal{R}_{(0,0),(0,1/2)}(\boldsymbol{u})$). Thus in order to determine the spectrum of an arbitrary operator $\mathcal{R}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}$ it is sufficient to calculate the spectrum of the $\mathcal{R}$ operator for the special values $\boldsymbol{\rho}_1=(0,0)$ and $\boldsymbol{\rho}_2=(0,\epsilon_2)$. The eigenfunctions of the $\mathcal{R}$ operator are fixed by the $SL(2,\mathbb{R})$ invariance of the latter. To find them one has to construct the operators separating the irreducible components in the tensor product of two representations of the continuous series, Eq. (2.8). The tensor product decomposition (2.8) contains the representations both of the discrete and continuous series. The operators $(\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(x_1,x_2,w))$ separating the representations of the discrete series, $\mathcal{D}^{\pm}_h$, are given in Eq. (A.9) [3]. Next, since the representations of the continuous series enter the tensor product decomposition Eq. (2.8) with double multiplicity, there is arbitrariness in the choice of the projectors. We defined the projectors to the continuous series, $\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(x_1,x_2,y)$, [4] (see Eq. (A.4)) in such way that they possess definite parity with respect to the interchange of the arguments x_1 and x_2. A direct check shows that namely such combinations diagonalizes the $\mathcal{R}-$operator. Introducing the notations for eigenvalues of the $\mathcal{R}-$operator on the corresponding eigenfunctions

$$\mathcal{R}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(u,\epsilon)\left(\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}\right)^{*} \;=\; R_{\rho,\varepsilon}(u,\epsilon)\left(\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}\right)^{*}\,, \tag{3.19}$$

$$\mathcal{R}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(u,\epsilon)\left(\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}\right)^{*} \;=\; R^{\pm}_{h}(u,\epsilon)\left(\Pi^{h,\pm}_{\boldsymbol{\rho}_1,\boldsymbol{\rho}_2}\right)^{*}\,, \tag{3.20}$$

one obtains after some calculations

[3] We remind that h takes integer or half-integer values.

[4] The parameter $\varepsilon=0,1/2$ marks different projectors.

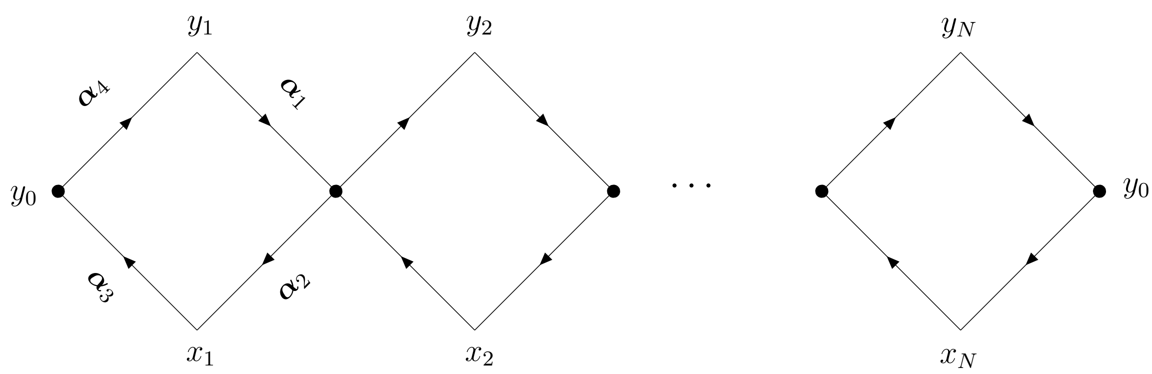

Figure 6: The diagrammatic representation of the kernel of the transfer matrix, Eq. (3.25). The indices $\boldsymbol{\alpha}_i$ of the lines on the k-th box are given by Eqs. (3.6) and (3.7) with $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_2 = \boldsymbol{\rho}_k$.

- $\epsilon_1 = \epsilon_2$

$$R_{\rho,\varepsilon}(u,\epsilon) = \frac{1}{\pi}\left[\sin(i\pi u) + (-1)^{2\epsilon+2\varepsilon}\sin(\pi s)\right]\Gamma(1-s-iu)\Gamma(s-iu)\,, \tag{3.21}$$

$$R_h^{\pm}(u,\epsilon) = (-1)^h\frac{\Gamma(h-iu)}{\Gamma(h+iu)}\,, \tag{3.22}$$

- $\epsilon_1 \neq \epsilon_2$

$$R_{\rho,\varepsilon}(u,\epsilon) = \frac{1}{\pi}\left[\cos(i\pi u) + (-1)^{2\epsilon+2\varepsilon}\cos(\pi s)\right]\Gamma(1-s-iu)\Gamma(s-iu)\,, \tag{3.23}$$

$$R_h^{\pm}(u,\epsilon) = (-1)^{h-1/2}\frac{\Gamma(h-iu)}{\Gamma(h+iu)}\,, \tag{3.24}$$

where $s = 1/2 + i\rho$.

3.2. Transfer matrices and Hamiltonians

Having obtained the solutions of the Yang-Baxter relation (3.1), one can construct an infinite set of the commuting $SL(2,\mathbb{R})$ invariant operators (transfer matrices)

$$\mathbb{T}_{\boldsymbol{\rho}_0}(\boldsymbol{u}) = \mathrm{tr}_{\boldsymbol{\rho}_0}\left(\mathcal{R}_{\boldsymbol{\rho}_0\boldsymbol{\rho}_1}(\boldsymbol{u})\ldots\mathcal{R}_{\boldsymbol{\rho}_0\boldsymbol{\rho}_N}(\boldsymbol{u})\right)\,, \tag{3.25}$$

where the trace is taken over the auxiliary space $V^{\boldsymbol{\rho}_0}$. They depend on the spectral parameter $\boldsymbol{u}$ and the spin of the auxiliary space, $\boldsymbol{\rho}_0$. The diagrammatic representation of the transfer matrix is shown in Fig. 6. Invoking the standard arguments [28], one finds that the transfer matrices form the family of the mutually commuting operators

$$\left[\mathbb{T}_{\boldsymbol{\rho}_0}(\boldsymbol{u}), \mathbb{T}_{\boldsymbol{\rho}_0'}(\boldsymbol{v})\right] = 0\,. \tag{3.26}$$

In the case of the homogeneous chain, (when all quantum spaces carry the same representation of the $SL(2,\mathbb{R})$ group), i.e. when $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = \ldots = \boldsymbol{\rho}_N \equiv \boldsymbol{\rho}_q$, it is possible to construct the

operator (Hamiltonian) which has two-particle structure. Indeed, choosing $\boldsymbol{\rho}_0 = \boldsymbol{\rho}_q$ and taking into account the property (3.18), one obtains

$$\mathcal{H}_N = i \left[\frac{d}{du} \ln \mathbb{T}_{\boldsymbol{\rho}_q}(\boldsymbol{u}) \right] \Bigg|_{\boldsymbol{u}=(0,0)} = H_{12} + \cdots + H_{N-1,N} + H_{N,1} \,, \tag{3.27}$$

where the two-particle Hamiltonian $H_{k,k+1}$ is given by

$$H_{12} \;=\; i\frac{d}{du} \ln \mathcal{R}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(\boldsymbol{u}) \Big|_{\boldsymbol{u}=\mathbf{0}} \,. \tag{3.28}$$

Introducing the notation $E^\varepsilon(\rho)$ for the eigenvalues of the pairwise Hamiltonian corresponding to the eigenfunctions $\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}$ [5] and $E^\pm(h)$ for those corresponding to the eigenfunctions $\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}$, one derives from Eqs. (3.21) the following expressions for the energies

$$E^\varepsilon(\rho) \;=\; \psi\left(\frac{1}{2} + i\rho\right) \;+\; \psi\left(\frac{1}{2} - i\rho\right) - (-1)^{2\varepsilon}\frac{\pi}{\cosh \pi\rho} \,, \tag{3.29}$$

$$E^\pm(h) \;=\; 2\psi(h) \,. \tag{3.30}$$

Herein $\psi(x)$, as usual, denotes the logarithmic derivative of the Euler $\Gamma-$function. Thus one sees that the energies corresponding to the discrete levels are double-degenerate. The separation between the continuous branches being large in region of small ρ, vanishes rapidly for large ρ. The corresponding dispersion curves are shown in Fig. 7. Note also, that although the eigenvalues of the two-particle Hamiltonian do not depend on the $\boldsymbol{\rho}_q$, this dependence reveals itself in the eigenfunctions. For the chains with the number of sites $N > 2$ the energies will explicitly depend on the values of $\boldsymbol{\rho}_q$.

Let us consider now the simplest among the transfer matrix — the auxiliary transfer matrix. It is given by the trace of the product of the Lax operators (3.13)

$$t_N(u) \;=\; \operatorname{tr}\left(L_1(u) \ldots L_N(u)\right) . \tag{3.31}$$

The auxiliary transfer matrix, $t_N(u)$, is the polynomial of degree N in u with operator valued coefficients

$$t_N(u) = 2u^N + \sum_{k=2}^{N} u^{N-k} q_k \,. \tag{3.32}$$

The operators q_k (integrals of motion) are differential operators in x_k. In particular,

$$q_2 \;=\; -\vec{S}^2 \;-\; N(\rho_q^2 + 1/4) \,, \tag{3.33}$$

where $\vec{S} = \vec{S}_1 + \ldots + \vec{S}_N$ is the operator of the total spin. The possible values of the total Casimir operator, $\vec{S}^2$, are restricted by the $SL(2,\mathbb{R})$ invariance, namely, $\vec{S}^2 = h(h-1)$ for the representations of the discrete series and $\vec{S}^2 = -1/4 - \rho^2$ for the representations of the continuous series.

Further, by virtue of Eq. (3.12) one derives that

$$\left[\vec{S}, t_N(u)\right] \;=\; [t_N(u), t_N(v)] \;=\; \left[t_N(u), \mathbb{T}_{\boldsymbol{\rho}_0}(\boldsymbol{v}))\right] \;=\; [t_N(u), \mathcal{H}_N] \;=\; 0 \,. \tag{3.34}$$

[5] Since $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2$, one gets that $\boldsymbol{\rho} = (\rho, 0)$.

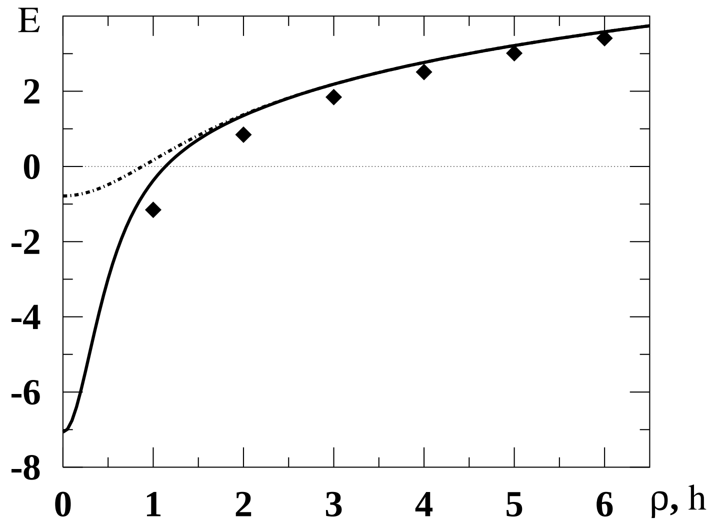

Figure 7: The dispersion curves for the two-particle Hamiltonian. The curves for $E^{(0)}(\rho)$ and $E^{(1/2)}(\rho)$ are shown by the solid line and the dot-dashed line, respectively. The discrete eigenvalues $E^{\pm}(h)\,(h = 1, 2, ..)$ are depicted by the black diamonds.

It follows from (3.34) that the auxiliary transfer matrix and Hamiltonian share a common set of eigenfunctions. Thus the eigenfunctions of the Hamiltonian can be labelled by the eigenvalues of the integrals of motion $q_2, \ldots, q_N$ and one of the $SL(2, \mathbb{R})$ generators, which is usually chosen to be S_-. However, in the case under consideration, these quantum numbers do not fix the eigenfunction completely. Indeed, as seen already for the two-site spin chain, the eigenfunctions of the Hamiltonian, corresponding to $E^0(\rho)$ and $E^{1/2}(\rho)$ (see Eq. (3.21)) have the same quantum numbers. Therefore, the spectrum of the transfer matrix is double degenerate, and fixing of the total momentum, $iS_- = p$, does not remove this degeneracy. Nevertheless, the number of eigenfunctions with identical integrals of motion, q_k, is finite. One can always resolve the degeneracy on this finite subspace by fixing an additional quantum number, for example the value of the quasimomentum θ. We remind that $e^{i\theta}$ is the eigenvalue of the operator of the cyclic permutation

$$\mathcal{P}\Psi(x_1, x_2, \ldots, x_N) = \Psi(x_2, \ldots, x_N, x_1)\,, \tag{3.35}$$

which commutes with both auxiliary transfer matrix and Hamiltonian, $[\mathcal{P}, t_N(u)] = [\mathcal{P}, \mathcal{H}_N] = 0$. Thus to uniquely determine the eigenstate, one should the specify the total momentum, p, and the following set of the quantum numbers, $\mathbf{q} = \{\theta, q_2, \ldots, q_N\}$.

4. Baxter $\mathbb{Q}-$operator

The solution of the eigenvalue problem for the Hamiltonian (3.27) can be obtained within the method of the Baxter $\mathbb{Q}-$operators [2]. The standard method of solving spin chain models, the Algebraic Bethe Ansatz (ABA) method [1, 28], is not applicable in the case under consideration

since the Hilbert space of the model does not have a normalizable lowest weight vector.

We remind that by definition the Baxter $\mathbb{Q}-$operator is the operator which acts on the Hilbert space of the model, depends on the spectral parameter and satisfies the following requirements

- Commutativity

$$[\mathbb{Q}(\boldsymbol{u}), \mathbb{Q}(\boldsymbol{v})] \;=\; [\mathbb{Q}(\boldsymbol{u}), t_N(v)] \;=\; [\mathbb{Q}(\boldsymbol{u}), \mathcal{H}_N] \;=\; 0\,, \tag{4.1}$$

- Baxter equation [6]

$$t_N(u)\,\mathbb{Q}(\boldsymbol{u}) \;=\; (u+is_q)^N\,\mathbb{Q}(\boldsymbol{u}+\mathbf{i}) \;+\; (u-is_q)^N\,\mathbb{Q}(\boldsymbol{u}-\mathbf{i})\,. \tag{4.2}$$

In the case under consideration, the spectral parameter, $\boldsymbol{u}$, as natural to expect, has the same nature as the spectral parameter of the $\mathcal{R}$ matrix, $\boldsymbol{u} = (u, \epsilon)$. We also used the notation $\mathbf{i} = (i, 1/2)$ and $s_q = 1/2 + i\rho_q$ in Eq. (4.2).

As usual we look for the operator $\mathbb{Q}$ in the form of an integral operator. Then Eq. (4.2) results in a particular differential equation on the kernel of the operator $\mathbb{Q}$. One can find the general solution of this equation using approach developed in [6, 7]. This is based on the invariance of the auxiliary transfer matrix (3.31) with respect to local rotations of the Lax operators

$$L_k(u) \to \tilde{L}_k(u) \;=\; M_k^{-1}\,L_k(u) M_{k+1}\,, \tag{4.3}$$

where M_k are arbitrary 2×2 nondegenerate matrices. Then one can show that the function $Y_{\boldsymbol{u}}(\vec{x},\vec{y})$,

$$Y_{\boldsymbol{u}}(\vec{x},\vec{y}) \;=\; \prod_{k=1}^{N} D_{\boldsymbol{\alpha}_u}(x_k - y_{k+1})\, D_{\boldsymbol{\beta}_u}(x_k - y_k)\,, \tag{4.4}$$

where

$$\boldsymbol{\alpha}_u \;=\; (s_q - iu, \epsilon+\epsilon_q)\,, \qquad \boldsymbol{\beta}_u = (s_q + iu, \epsilon)\,, \qquad \boldsymbol{\alpha}_u + \boldsymbol{\beta}_u = (2s_q, \epsilon_q)\,, \tag{4.5}$$

satisfies the following equation

$$t_N(u)\,Y_{\boldsymbol{u}}(\vec{x},\vec{y}) \;=\; (u+is)^N\,Y_{\boldsymbol{u}+\mathbf{i}}(\vec{x},\vec{y}) + (u-is)^N\,Y_{\boldsymbol{u}-\mathbf{i}}(\vec{x},\vec{y}) \tag{4.6}$$

at arbitrary values of variables $y_1,\ldots,y_N$. This means that the convolution of the function $Y_{\boldsymbol{u}}(\vec{x},\vec{y})$ with an arbitrary function of $Z(\vec{y})$ also satisfies Eq. (4.2). The details of the derivation of Eq.(4.6) can be found in Ref. [11] where the similar case of the $SL(2,\mathbb{C})$ spin chain was considered. Let us only note here that one can consider the function $Y_{\boldsymbol{u}}(\vec{x},\vec{y})$ as an array of two functions which depends on the complex parameter u only, $Y_{\boldsymbol{u}} = \big(Y_{(u,0)}, Y_{(u,1/2)}\big)$. Then Eq. (4.6) takes form of the matrix finite-difference equation.

Thus the kernel of the Baxter operator can be written in the form

$$Q_{\boldsymbol{u}}(\vec{x},\vec{x}') \;=\; \int\prod_{k=1}^{N} dy_k\, Y_{\boldsymbol{u}}(\vec{x},\vec{y})\, Z(\vec{y},\vec{x}')\,, \tag{4.7}$$

[6]Similarly to the Yang-Baxter equation (3.1) this equation contains two finite-difference equations which entangle the Baxter operators $\mathbb{Q}(u,0)$ and $\mathbb{Q}(u,1/2)$.

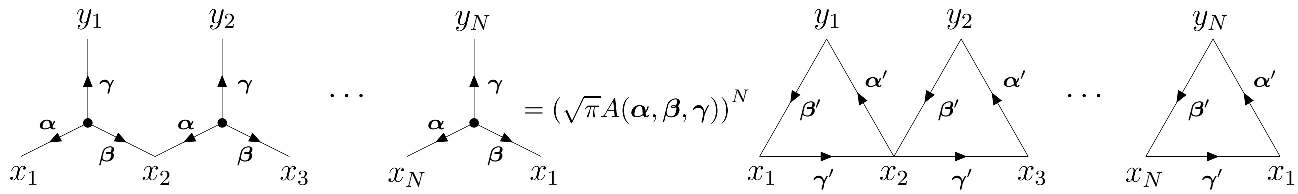

Figure 8: The kernel of the $\mathbb{Q}$-operator in two equivalent representations. The indices are explained in the text; $\boldsymbol{x}' = \mathbf{1} - \boldsymbol{x}$.

where the function $Z(\vec{y}, \vec{x}')$ does not depend on the spectral parameter $\boldsymbol{u}$. The restrictions on this function can be deduced from the requirement of the commutativity of the Baxter $\mathbb{Q}-$operator and the auxiliary transfer matrix. To this end let us note that the function $Y_{\boldsymbol{u}}(\vec{x}, \vec{y})$ possesses the following property

$$Y_{\boldsymbol{u}}(\vec{x}, \vec{y}) \;=\; (-1)^{2N\epsilon_q}\, \mathcal{P}\, Y_{\boldsymbol{u}'}(\vec{y}, \vec{x}), \tag{4.8}$$

where $\boldsymbol{u}' = (-u, \epsilon + \epsilon_q)$ and $\mathcal{P}$ is the operator of the cyclic permutation (3.35). Then taking into account Eq. (4.6) and the invariance of the transfer matrix under cyclic permutations, $[\mathcal{P}, t_N(u)] = 0$, one derives

$$t_N(u, \vec{S}(\rho_q, x))\, Y_{\boldsymbol{u}}(\vec{x}, \vec{y}) \;=\; (-1)^N\, t_N(-u, \vec{S}(\rho_q, y))\, Y_{\boldsymbol{u}}(\vec{x}, \vec{y}) \;=\; t_N(u, -\vec{S}(\rho_q, y))\, Y_{\boldsymbol{u}}(\vec{x}, \vec{y})\,. \tag{4.9}$$

Here we indicated explicitly that the transfer matrix $t_N(u)$ is expressed in terms of the differential operators (2.6) acting on x or y-coordinates. Then, noticing that the integration by parts results in the change $\rho_q \to -\rho_q$ in the generators $\int dy\, [S(\rho_q, y)\Phi(y)]\, \Psi(y) = \int dy \Phi(y)\, [S(-\rho_q, y)\Psi(y)]$ and taking advantage of Eq. (4.9), one derives that the requirements $[t_N(u), \mathbb{Q}(\boldsymbol{u})] = 0$ gives the following equation on the function $Z(\vec{x}, \vec{y})$

$$t_N(u, S(-\rho_q, y))\, Z(\vec{y}, \vec{x}') \;=\; Z(\vec{y}, \vec{x}')\, t_N(u, S(\rho_q, x'))\,. \tag{4.10}$$

The simplest solution to this equation corresponds to the operator Z intertwining the generators $\vec{S}_k(-\rho_q)$ and $\vec{S}_k(\rho_q)$. Thus one can choose the operator Z to be proportional to the product $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \ldots \otimes \mathcal{M}_N$, where the intertwining operators $\mathcal{M}_k$ are defined in (2.3). For later convenience we put

$$Z(\vec{x}, \vec{y}) \;=\; \prod_{k=1}^{N} D_{\boldsymbol{\gamma}}(y_{k-1} - x_k)\,. \tag{4.11}$$

with $\boldsymbol{\gamma} = (2 - 2s_q, \epsilon_q)$.

As usual, we visualize the kernel of the $\mathbb{Q}-$operator as a Feynman diagram, which is shown in the lhs of Fig. 8. In the rhs of Fig. 8, we give the equivalent representation of the kernel (4.7) which can be obtained from the diagram on the lhs with the help of the star-triangle relation (3.10). The graphical representation of the kernel is very convenient for the analysis of the properties of the Baxter $\mathbb{Q}-$operator. As an example we prove the first relation in (4.1), namely $[\mathbb{Q}(\boldsymbol{u}), \mathbb{Q}(\boldsymbol{v})] = 0$.

The diagrammatic representation of the integral kernel of the operator $\mathbb{Q}(\boldsymbol{u})\, \mathbb{Q}(\boldsymbol{v})$ is shown in Fig. 9. There we represent the kernel of the operator $\mathbb{Q}(\boldsymbol{v})$ by the diagram shown in the lhs of Fig. 8, while for the operator $\mathbb{Q}(\boldsymbol{u})$ we choose the alternative representation given by the diagram in the rhs of Fig. 8. We also inserted two propagators with opposite indices in the central

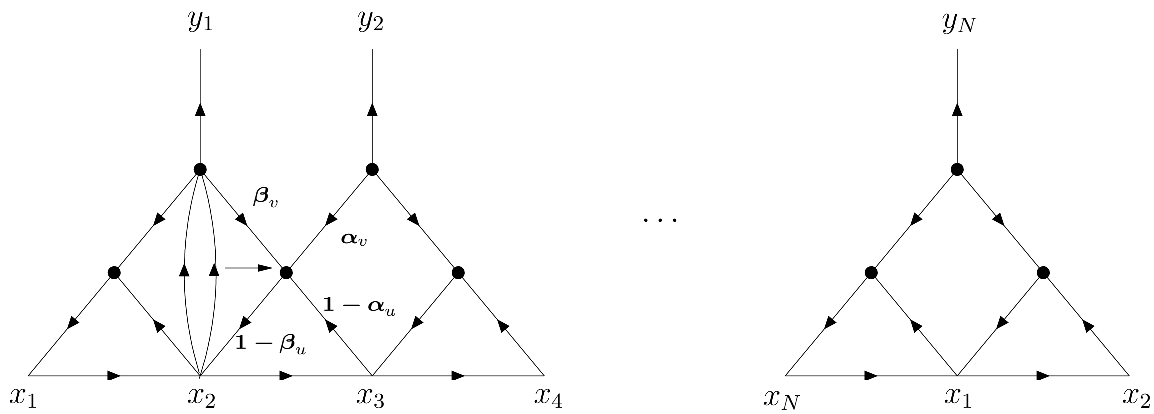

Figure 9: The diagrammatic representation of the product $\mathbb{Q}(u)\mathbb{Q}(v)$.

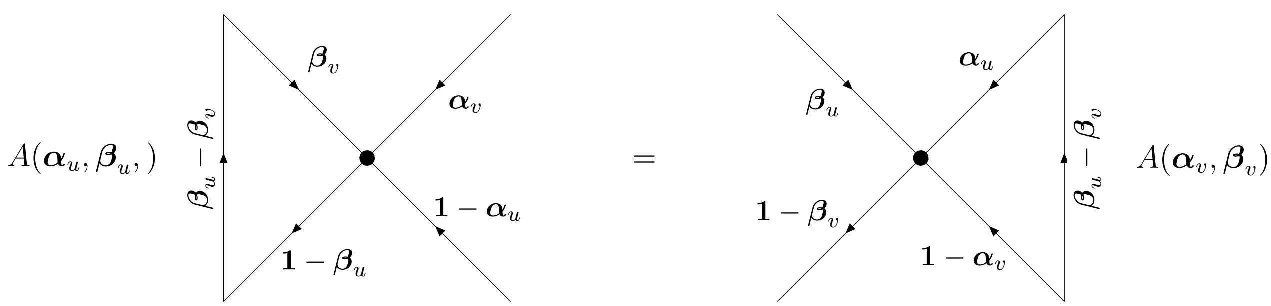

Figure 10: The diagrammatic representation of the exchange relation. The indices take the special values $\boldsymbol{\alpha}_x = (s+ix, \epsilon_x+\epsilon)$ and $\boldsymbol{\beta}_x = (s-ix, \epsilon)$, where $s = 1/2+i\rho$, $(\boldsymbol{\alpha}_u+\boldsymbol{\beta}_u = \boldsymbol{\alpha}_v+\boldsymbol{\beta}_v = (2s, \epsilon))$.

rhombus in Fig. 9. Since $D_{\boldsymbol{\sigma}}(x)\, D_{-\boldsymbol{\sigma}}(x) = 1$, such an insertion does not alter the value of the diagram.

The proof is based on the use of an additional diagrammatic identity– exchange relation– which is shown in Fig. 10. This identity can be easily derived with the help of the star-triangle relation (3.10). Further, to prove the commutativity of the Baxter operators we choose the index of the inserted propagator to be, $\boldsymbol{\sigma} = \boldsymbol{\beta}_u - \boldsymbol{\beta}_v$. Then one can use the exchange relation to move this inserted propagator, $D_{\boldsymbol{\sigma}}$, to the right. The displacing of the propagator $D_{\boldsymbol{\sigma}}$ from left to right of the cross subdiagram alters the indices of the propagators and produces some scalar factor in the way shown in Fig. 10. After repeating this operation N times the propagator resumes its initial position and annihilates the propagator $D_{-\boldsymbol{\sigma}}$. Therefore the resulting diagram will have the same form as that in Fig. 9. Examing the indices of the propagators and the scalar factor one finds that they correspond to the diagram for the operator $\mathbb{Q}(\boldsymbol{v})\,\mathbb{Q}(\boldsymbol{u})$, if the representation in the lhs of Fig. 8 is used for $\mathbb{Q}(\boldsymbol{u})$ and for operator $\mathbb{Q}(\boldsymbol{v})$ those in the rhs of the same figure. Thus one concludes that

$$\mathbb{Q}(\boldsymbol{u})\,\mathbb{Q}(\boldsymbol{v}) \;=\; \mathbb{Q}(\boldsymbol{v})\,\mathbb{Q}(\boldsymbol{u})\,. \tag{4.12}$$

The Eqs. (4.4), (4.11) together with (4.7) define the Baxter $\mathbb{Q}-$operator which satisfies Eqs. (4.1), (4.2). (Strictly speaking we did not show yet that the Baxter operator commutes with the Hamiltonian but will do it later on.)

4.1. Properties of $\mathbb{Q}-$operator

Let us examine the Baxter $\mathbb{Q}-$operator for the special values of the spectral parameter such that indices $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ become equal zero. Using the diagrammatic representation of the latter given in Fig. 8, one finds that for $\boldsymbol{u} = (is_q, 0)$ ($\boldsymbol{\beta} = \mathbf{0}$) the lines with the index $\boldsymbol{\beta}$ disappear and the integral over centers of the star diagrams can be easily calculated with help of Eq. (3.9). A short calculation gives

$$\mathbb{Q}(is_q, 0) \;=\; c(\boldsymbol{\rho}_q)^N \, \mathbb{I}\,, \tag{4.13}$$

where $\mathbb{I}$ is the unit operator and the normalization constant is given by

$$c(\boldsymbol{\rho}) \;=\; \pi |A(1+2i\rho,\epsilon)|^2 = \frac{\pi}{\rho} \tanh^{4\epsilon-1}(\pi\rho)\,. \tag{4.14}$$

Repeating this calculation for $\boldsymbol{u} = (-is_q, \epsilon_q)$ ($\boldsymbol{\alpha} = \mathbf{0}$), one finds

$$\mathbb{Q}(-is_q, \epsilon_q) \;=\; c(\boldsymbol{\rho}_q)^N \, \mathcal{P}\,, \tag{4.15}$$

where $\mathcal{P}$ is the operator of the cyclic permutations (3.35). Therefore, the operator $\mathcal{P}$ can be expressed as the ratio of the Baxter operator at two special points

$$\mathcal{P} \;=\; \mathbb{Q}(-is_q, \epsilon_q)/\mathbb{Q}(is_q, 0)\,. \tag{4.16}$$

Next, using the diagrammatic representation for the Baxter $\mathbb{Q}-$operator (Fig. 8) and for the transfer matrix (Fig. 6) one finds after some algebra

$$T_{\boldsymbol{\rho}_0}(\boldsymbol{u}) \;=\; \chi(\boldsymbol{u}, \boldsymbol{\rho}_0, \boldsymbol{\rho}_q)\, [\mathbb{Q}(u^* - is_0, \epsilon + \epsilon_0)]^\dagger\, \mathbb{Q}(u + is_0, \epsilon) = \tag{4.17}$$

$$= \;\bar{\chi}(\boldsymbol{u}, \boldsymbol{\rho}_0, \boldsymbol{\rho}_q)\, \mathbb{Q}(u + i(1-s_0), \epsilon + \epsilon_0)\, [\mathbb{Q}(u^* - i(1-s_0), \epsilon)]^\dagger\,. \tag{4.18}$$

The normalization factors are given by the following expressions

$$\chi(\boldsymbol{u}, \boldsymbol{\rho}_0, \boldsymbol{\rho}_q) \;=\; e^{i\pi(\epsilon_0+\epsilon_q)N} \left[\frac{2^{-2iu}}{\pi}\right]^N c(\boldsymbol{\rho}_q)^{-N} \tag{4.19}$$
$$\times \left[A(1 + i\rho_q + i\rho_0 - iu, \epsilon + \epsilon_q)\, A(1 - i\rho_q - i\rho_0 - iu, \epsilon + \epsilon_0)\right]^{-N}\,,$$

$$\bar{\chi}(\boldsymbol{u}, \boldsymbol{\rho}_0, \boldsymbol{\rho}_q) \;=\; (-1)^{2(\epsilon_0+\epsilon_q)N}\, c(\boldsymbol{\rho}_q)^{-N}\, r_{\boldsymbol{\rho}_0\boldsymbol{\rho}_q}(\boldsymbol{u})^N\,, \tag{4.20}$$

where $r_{\boldsymbol{\rho}_0\boldsymbol{\rho}_q}(\boldsymbol{u})$ is defined in Eq. (3.16). Thus an arbitrary transfer matrix (3.25) can be expressed as the product of two Baxter $\mathbb{Q}-$operators at special values of the spectral parameter.

Using the diagrammatic representation, it is also straightforward to show that the operators $\mathbb{Q}$ and $\mathbb{Q}^\dagger$ commute at arbitrary values of the spectral parameters. Moreover, one can deduce the following identity

$$\left[(-1)^{2\epsilon_v} A(\boldsymbol{\alpha}_v) A(\boldsymbol{\beta}_v)\right]^N \mathbb{Q}(\boldsymbol{u})\, [\mathbb{Q}(\boldsymbol{v^*})]^\dagger \;=\; \left[(-1)^{2\epsilon_u} A(\boldsymbol{\alpha}_u) A(\boldsymbol{\beta}_u)\right]^N \mathbb{Q}(\boldsymbol{v})\, [\mathbb{Q}(\boldsymbol{u^*})]^\dagger\,, \tag{4.21}$$

where $\boldsymbol{\alpha}_u$, $\boldsymbol{\beta}_u$ are defined in (4.4).

Having put $\boldsymbol{\rho}_0 = \boldsymbol{\rho}_q$ in Eq. (4.17) and taking into account (3.27), one obtains the following expression for the Hamiltonian $\mathcal{H}_N$ in terms of the Baxter $\mathbb{Q}-$operator

$$\mathcal{H}_N \;=\; i\,\frac{d}{du} \ln \mathbb{Q}(is_q + u, 0)\Big|_{u=0} \;+\; i\,\frac{d}{du} \ln \mathbb{Q}(-is_q + u, \epsilon_q)^\dagger\Big|_{u=0} \;+\; \mathcal{E}_N$$

$$= \;i\,\frac{d}{du} \ln u^N \mathbb{Q}(is_q^* + u, \epsilon_q)\Big|_{u=0} \;+\; i\,\frac{d}{du} \ln u^N \mathbb{Q}(-is_q^* + u, 0)^\dagger\Big|_{u=0} \;+\; \overline{\mathcal{E}}_N\,. \tag{4.22}$$

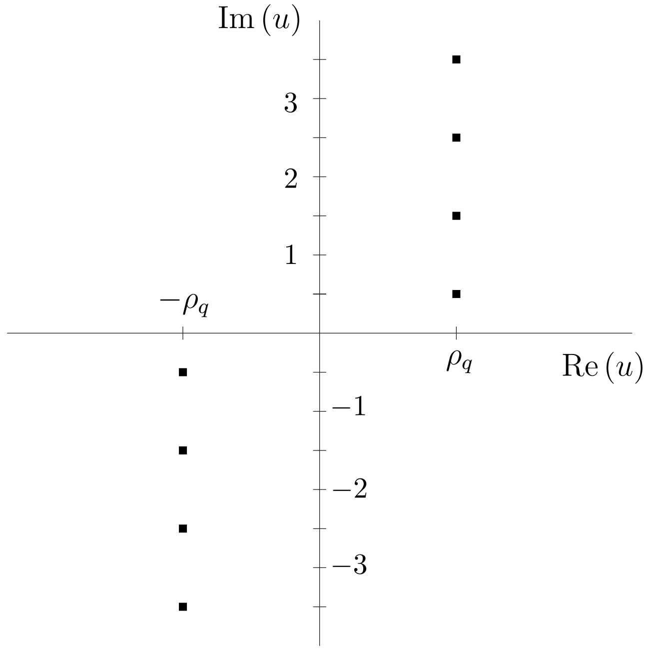

Figure 11: Distribution of the poles of the Baxter $\mathbb{Q}$-operator in the complex u-plane for $\epsilon = \epsilon_q = 0$.

where the additive constants $\mathcal{E}_N$ and $\overline{\mathcal{E}}_N$ are

$$\mathcal{E}_N \;=\; N\left[\psi(2\epsilon_q + 2i\rho_q) \;+\; \psi(2\epsilon_q - 2i\rho_q)\right], \qquad\qquad \overline{\mathcal{E}}_N \;=\; 2N\,\psi(1)\,. \tag{4.23}$$

Thus the Baxter $\mathbb{Q}-$operator commutes with the Hamiltonian $\mathcal{H}_N$ (3.27). Evidently, the relations (4.22), (4.16), (4.17) hold for the eigenvalues of the corresponding operators as well. Thus, the knowledge of the eigenvalue of the Baxter $\mathbb{Q}-$operator allows one to restore the eigenvalues of all other operators in question. The remarkable property of the Baxter $\mathbb{Q}-$operator is that its eigenvalues can be determined without solving the eigenvalue problem. Indeed, since the Baxter $\mathbb{Q}-$operator satisfies Eq. (4.2), the same equation holds for its eigenvalues. Therefore solving this equation in the appropriate class of functions one can determine all eigenvalues of the $\mathbb{Q}-$operator. In order to be an eigenvalue of the Baxter $\mathbb{Q}-$operator the solution of Eq. (4.2) should satisfy the additional conditions. To find them we shall study in the next section the analytic properties of the eigenvalues of the $\mathbb{Q}-$operator as the function of u.

4.2. Analytic structure and asymptotic

We shall denote the eigenvalue of the Baxter $\mathbb{Q}-$operator corresponding to the eigenfunction $\Psi_{q_2,\ldots,q_N}(\vec{x})$ as $Q_{\mathbf{q}}(\boldsymbol{u}) \equiv Q_{\mathbf{q}}(u,\epsilon)$ and study its properties as a function of the complex variable u. It follows directly from the definition (4.4) and (4.7) that the Baxter $\mathbb{Q}-$operator is well defined as the integral operator in the strip $-1/2 < iu < 1/2$. To continue it to the whole complex plane, let us apply the Baxter operator, $\mathbb{Q}(\boldsymbol{u})$, to a test function $\Psi(\vec{x})$. Then it can be seen that the resulting function $\Phi_{\boldsymbol{u}}(\vec{x}) = [\mathbb{Q}(\boldsymbol{u})\Psi](\vec{x})$ as a function of u admits continuation to the whole complex plane except for the points where the indices α and β take positive integer values,

namely $u_n^- = is_q - in$, $(\beta = n > 0)$ and $u_n^+ = -is_q + in$ $(\alpha = n > 0)$. The structure of the singularities can be easily established using the representation of the Baxter $\mathbb{Q}-$operator given in the rhs of Fig. 8. Since the indices of the propagators α' and β' are not positive integers while α or β are, the corresponding integrals define a regular function of u at these points and all singularities are contained in the prefactor $A(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})^N$. The straightforward analysis shows that the Baxter $\mathbb{Q}-$operator has poles of the order N in the upper and lower half-planes in the following points

- Upper half-plane

$$\begin{aligned} &\epsilon = \epsilon_q, \quad u_n^+ = \rho_q + 2in + \tfrac{i}{2}, \quad n \ge 0 \\ &\epsilon \neq \epsilon_q, \quad u_n^+ = \rho_q + 2in, \qquad\quad n \ge 1 \end{aligned} \tag{4.24}$$

- Lower half-plane

$$\begin{aligned} &\epsilon = 0, \qquad u_n^- = -\rho_q - 2in - \tfrac{i}{2}, \quad n \ge 0 \\ &\epsilon = 1/2, \quad u_n^- = -\rho_q - 2in, \qquad\quad n \ge 1 \end{aligned} \tag{4.25}$$

Thus the eigenvalue of the Baxter $\mathbb{Q}$-operator is a meromorphic function of the variable u with poles of the order N at the points $u_n^\pm(\epsilon)$, Eqs. (4.24) and (4.25). The positions of the poles in the u plane are schematically shown in Fig. 11. Note also that the equation in the second line of (4.22) gives the energy as the ratio of the residues at the leading (of order N) and the subleading (of order $N-1$) poles at $u_0^\pm = \pm(\rho_q + i/2)$.

Since the solutions of the Baxter equation (4.2) are determined up to multiplication by the periodic function $f(\boldsymbol{u}+\mathbf{i}) = f(\boldsymbol{u})$ one needs additional information to fix this arbitrariness. This can be obtained by the study of the asymptotic behaviour of the eigenvalues at large u. First of all, let us notice that the transformation properties of the eigenfunction under $SL(2,\mathbb{R})$ rotation is determined by the eigenvalue of the charge q_2 or the total Casimir operator, $\vec{S}^2 \equiv \left(\vec{S}_1 + \ldots + \vec{S}_N\right)^2$ (see Eq. (3.33)). Namely, $\vec{S}^2 = -1/4 - \rho^2$ if the eigenfunction transforms according the irreducible representation of the principal series, $T^{(\rho,\epsilon)}$, and $\vec{S}^2 = h(h-1)$ if it transforms according to the discrete series representation $\mathcal{D}_h^\pm$. Thus depending on the value of the Casimir operator, the eigenfunction can be represented as follows

$$\Psi^c_{\mathbf{q}}(x_1,\ldots,x_N) = \int_{-\infty}^{\infty} dx_0\, \Pi^{(\rho,\epsilon)}_{q_3,\ldots,q_N}(x_1,\ldots,x_N|x_0)\, f^{(\rho,\epsilon)}(x_0)\,, \tag{4.26}$$

$$\Psi^d_{\mathbf{q}}(x_1,\ldots,x_N) = \int \mathcal{D}_\pm w\, \Pi^{h,\pm}_{q_3,\ldots,q_N}(x_1,\ldots,x_N|\bar{w})\, \phi_h^\pm(w)\,, \tag{4.27}$$

where the integration measure in (4.27) is defined in Eq. (A.3). The projectors $\Pi^{(\rho,\epsilon)}$, $\Pi^{h,\pm}$ intertwine the representations $(\otimes T^{\boldsymbol{\rho_q}})^N$ and $T^{(\rho,\epsilon)}$, $\mathcal{D}_h^\pm$, respectively. They have the following transformation properties

$$\Pi^{(\rho,\epsilon)}_{q_3,\ldots,q_N}(\vec{x}|x_0) = \frac{\sigma_\epsilon(cx_0+d)}{|cx_0+d|^{2-2s}} \prod_{k=1}^N \frac{\sigma_{\epsilon_q}(cx_k+d)}{|cx_k+d|^{2s_q}}\, \Pi^{(\rho,\epsilon)}_{q_3,\ldots,q_N}(\vec{x'}|x_0')\,, \tag{4.28}$$

$$\Pi^{h,\pm}_{q_3,\ldots,q_N}(\vec{x}|\bar{w}) = \frac{1}{(c\bar{w}+d)^{2h}} \prod_{k=1}^N \frac{\sigma_{\epsilon_q}(cx_k+d)}{|cx_k+d|^{2s_q}}\, \Pi^{h,\pm}_{q_3,\ldots,q_N}(\vec{x'}|\bar{w}')\,, \tag{4.29}$$

where $x'_k = (ax_k + b)/(cx_k + d)$ and $\bar{w}' = (a\bar{w} + b)/(c\bar{w} + d)$; a, b, c, d are real and $ad - bc = 1$. In particular, one finds that the projectors are invariant under the simultaneous shift of all arguments by a real number and are transformed under scale transformations as

$$\Pi^{(\rho,\epsilon)}_{q_3,\ldots,q_N}(\lambda x_1, \ldots, \lambda x_N|\lambda x_0) = \lambda^{-1+s-Ns_q}\Pi^{(\rho,\epsilon)}_{q_3,\ldots,q_N}(x_1, \ldots, x_N|x_0)\,, \tag{4.30}$$

$$\Pi^{h.\pm}_{q_3,\ldots,q_N}(\lambda x_1, \ldots, \lambda x_N|\lambda \bar{w}) = \lambda^{-h-Ns_q}\Pi^{h,\pm}_{q_3,\ldots,q_N}(x_1, \ldots, x_N|\bar{w})\,. \tag{4.31}$$

Applying the $\mathbb{Q}-$operator in the form (4.7) to the eigenfunction (4.26), one finds that the leading contributions at $u \to \infty$ comes from two integration regions over $\vec{y}$

$$(\mathrm{I}): \quad |y_k| = \mathcal{O}(u), \qquad (\mathrm{II}): \quad y_k - y_{k+1} = \mathcal{O}(1/u)\,. \tag{4.32}$$

Next, let us notice that any function $f^{(\rho,\epsilon)}(x)$ can be represented as the transformation of the function $f_0(x) = 1$, $f^{(\rho,\epsilon)}(x) = \int Dg\phi(g)T^{(\rho,\epsilon)}(g)f_0(x)$, where the integral is taken over the group. Then taking into account the invariance of the Baxter $\mathbb{Q}-$operator with respect to the $SL(2,\mathbb{R})$ transformations and the properties of the projector $\Pi^{(\rho,\epsilon)}_{q_3,\ldots,q_N}$ (4.28), one can derive (see Ref. [11] for details)

$$Q_{\mathbf{q}}(u,\epsilon) \overset{u\to\infty}{=} \left(A_I\, u^{s-Ns_q} \;+\; A_{II}\, u^{1-s-Ns_q}\right)\,[1 + \mathcal{O}(1/u)]\;. \tag{4.33}$$

The constants A_I and A_{II} depend on the integrals of motion q_k, and we remind that $s = 1/2 + i\rho$, $s_q = 1/2 + \rho_q$.

In the case of the eigenfunctions of the discrete spectrum (4.27), a careful analysis shows that the contributions coming from the regions (I) and (II) (4.32) are of the same order

$$Q_{\mathbf{q}}(u,\epsilon) \overset{u\to\infty}{=} C\, u^{1-h-Ns_q}\,[1 + \mathcal{O}(1/u)]\;. \tag{4.34}$$

Let us also note that since the kernels of the operators $\mathbb{Q}(u,0)$ and $\mathbb{Q}(u,1/2)$ coincide in the regions (I) and (II), the difference $Q_{\mathbf{q}}(u,0) - Q_{\mathbf{q}}(u,1/2)$ vanishes faster than any degree of $1/u$, i.e.

$$Q_{\mathbf{q}}(u,0) - Q_{\mathbf{q}}(u,1/2) \overset{u\to\pm\infty}{=} \mathcal{O}(e^{-\kappa|u|})\,, \tag{4.35}$$

where κ is some constant.

Thus the solution of the Baxter equation (4.2) corresponds to some eigenvalue of the Baxter $\mathbb{Q}$-operator only if it has the proper pole structure (4.24), (4.25) and proper asymptotic (4.33), (4.34) at $u \to \pm\infty$.

5. Separation of Variables

In this section we explicitly construct the representation of the Separated Variables for the model in question. Namely, we shall obtain the following integral representation for the eigenfunctions of the model

$$\Psi_{\mathbf{q}}(\vec{x}) = \frac{1}{2\pi}\int dp \int \prod_{k=1}^{N-1} \mathcal{D}\boldsymbol{u}_k\, \mu_N(\vec{\boldsymbol{u}})\, \Phi_{\mathbf{q}}(p, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1}) U_{p,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N-1}}(\vec{x})\,, \tag{5.1}$$

where

$$\int \mathcal{D}\boldsymbol{u}_k \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} du_k \sum_{\epsilon_k=0,1/2}\,, \tag{5.2}$$

$\mu_N(\vec{\boldsymbol{u}})$ is the Sklyanin measure (see Eq. (5.12)), $\Phi_{\mathbf{q}}(p, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1})$ is the eigenfunction in the SoV representation (5.16) and $U_{p,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N-1}}(x_1, \ldots, x_N)$ is the transition kernel to the SoV representation (5.5).

In Sklyanin's approach [4] the functions $U_{p,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N-1}}(x_1, \ldots, x_N)$ are identified as the eigenfunctions of the operator $B_N(u)$ — an off diagonal matrix element of the monodromy matrix. Provided that the spectrum of the operator $B_N(u)$ is non-degenerate, one can derive that the eigenfunction in the SoV representation ($\Phi_{\mathbf{q}}(p, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1})$) satisfies the Baxter equation in each variable. In the case under consideration this condition is not fulfilled. Moreover, the hermitian operator $B_N(u)$ admits different non-equivalent self-adjoint extensions and one has to choose the correct one. Although these problems can be overcome, we shall construct the transition kernel $U_{p,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N-1}}(x_1, \ldots, x_N)$ following another approach. It was conjectured in Ref. [29] that the transition kernel to the SoV representation can be related to the kernel corresponding to the product of the Baxter $\mathbb{Q}-$operators. The representation of such type is known now for a number of spin chain models [11, 12, 15]. We suggest the following ansatz for the transition kernel

$$U_{p,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N-1}}(\vec{x}) = c_N(p) \int \prod_{k=1}^{N} dy_k\, e^{ipy_1}\; [\mathbb{Q}(\boldsymbol{u}_1) \ldots \mathbb{Q}(\boldsymbol{u}_{N-1})]\,(x_1, \ldots, x_N|y_1, \ldots, y_N) \tag{5.3}$$

and show that so constructed kernel possesses all necessary properties. For convenience we put

$$c_N(p) = |p|^{-N+1/2}\, c(\boldsymbol{\rho}_q)^{-N(N-1)/2}\,,$$

where $c(\boldsymbol{\rho}_q)$ is defined in (4.14).

First of all let us notice that, as follows from the properties of the Baxter $\mathbb{Q}$-operator, the kernel (5.3) is a symmetric function of the variables $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1}$. Second, it has a definite momentum $i(S_-^{(1)} + \cdots + S_-^{(N)})U_{p,\vec{\boldsymbol{u}}}(\vec{x}) = pU_{p,\vec{\boldsymbol{u}}}(\vec{x})$ and, at last, it satisfies the Baxter equation (4.2) in each variable $\boldsymbol{u}_k$.

Further, we shall show that for real $\boldsymbol{u}_k$, $k = 1, \ldots, N-1$, the functions $U_{p,\vec{\boldsymbol{u}}}(\vec{x})$ are mutually orthogonal

$$\int \prod_{k=1}^{N} dx_k\, (U_{q,\vec{\boldsymbol{v}}}(\vec{x}))^*\, U_{p,\vec{\boldsymbol{u}}}(\vec{x}) \;=\; 2\pi\delta(p-q)\,\delta(\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}})\,\frac{\mu_N^{-1}(\vec{\boldsymbol{u}})}{(N-1)!}\,, \tag{5.4}$$

where

$$\delta(\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}) = \sum_S \delta(\boldsymbol{u}_1 - \boldsymbol{v}_{i_1}) \ldots \delta(\boldsymbol{u}_{N-1} - \boldsymbol{v}_{i_{N-1}})\,,$$

the sum goes over all permutations and $\delta(\boldsymbol{u} - \boldsymbol{v}) = (2\pi)^2\delta_{\epsilon_u\epsilon_v}\delta(u - v)$. To prove (5.4) it is convenient to represent the kernel (5.3) in another form. Namely, let us use the diagrammatic representation for the Baxter $\mathbb{Q}$-operator shown in the rhs of Fig. 8. The diagram for the product of $N-1$ $\mathbb{Q}$-operators consists of $N-1$ such rows. We shall assume that each triangle in Fig. 8 accompanied by the corresponding scalar factor $\sqrt{\pi}A(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. According to Eq. (5.3) one has to perform the integration over variables $y_2, \ldots, y_N$ which correspond to the upper vertices of the $N-1$ triangles in the upper row. Since only two propagators are attached to each vertex in question, one can use the chain relation, Eq. (3.9). The resulting propagators have the index $\boldsymbol{\gamma} = (1 - 2s_q, \epsilon_q)$ and cancel the $N-1$ horizontal lines in the upper row of the diagram. Thus after the integration over its upper vertex, the triangle disappears producing the scalar factor $c(\boldsymbol{\rho}_q) = \pi|A((1 + 2i\rho_q, \epsilon_q))|^2$. As a consequence one obtains a new diagram in which $N-1$

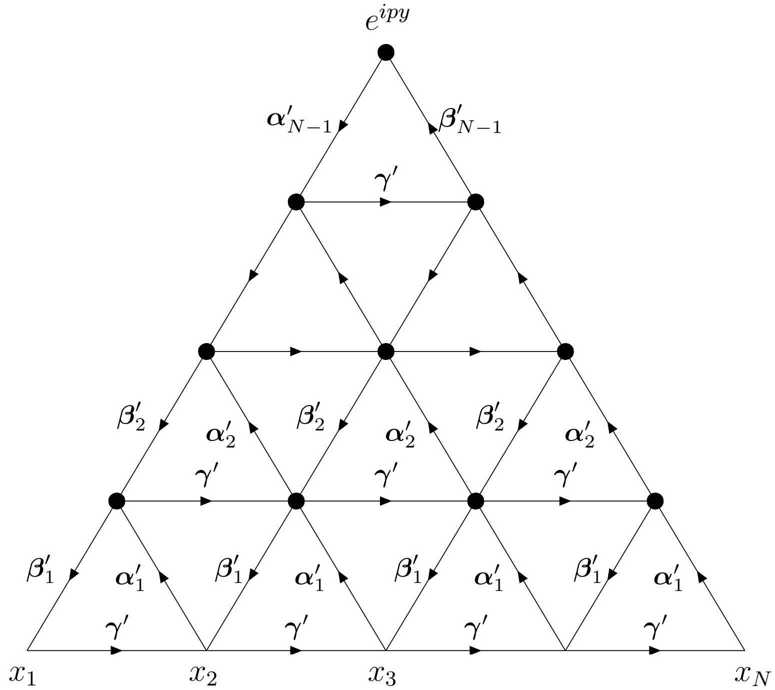

Figure 12: The graphical representation of the transition kernel, Eq. (5.5).

triangles in the upper row are eliminated and the additional scalar factor $c(\boldsymbol{\rho}_q)^{N-1}$ is acquired. Repeating this operation $N-1$ times one derives that the kernel (5.3) is represented by the diagram shown in Fig. 12 with the accompanying additional factor $c(\boldsymbol{\rho}_q)^{N(N-1)/2}$. Therefore the function (5.3) can be represented in the form

$$U_{p,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N-1}}(\vec{x}) \;=\; |p|^{-N+1/2} \int dy\, e^{ipy}\, \left[\Lambda_N(\boldsymbol{u}_1)\ldots\Lambda_2(\boldsymbol{u}_{N-1})\right](x_1,\ldots,x_N|\, y), \tag{5.5}$$

where the function $\Lambda_k(x_1,\ldots,x_k|y_1,\ldots,y_{k-1})(\boldsymbol{u}_{N+1-k})$ corresponds to the $k-1$–th row (from the top) of triangles in the diagram shown in Fig. 12. We remind here that each triangle is accompanied by the corresponding scalar factor.

The representation (5.5) is convenient for the calculation of the scalar product (5.4). Using the representation (5.5) for both $U-$functions, one finds that the scalar product (5.4) takes the form (up to some prefactor)

$$\int dy'\, e^{-iqy'} \int dy\, e^{ipy} \left[\Lambda_2(\boldsymbol{v}_1)^\dagger \ldots \Lambda_N(\boldsymbol{v}_{N-1})^\dagger\, \Lambda_N(\boldsymbol{u}_1)\ldots\Lambda_2(\boldsymbol{u}_{N-1})\right](y'|\, y)\,. \tag{5.6}$$

It can be easily checked that for real $\boldsymbol{u}$, $\boldsymbol{v}$, $(\boldsymbol{u} \neq \boldsymbol{v})$, the $\Lambda-$operators satisfy the following (exchange) relation

$$\Lambda_k(\boldsymbol{v})^\dagger\, \Lambda_k(\boldsymbol{u}) \;=\; c(\boldsymbol{\rho}_q)\, \varphi(\boldsymbol{u},\boldsymbol{v})\, \Lambda_{k-1}(\boldsymbol{u})\, \Lambda_{k-1}(\boldsymbol{v})^\dagger \tag{5.7}$$

with

$$\varphi(\boldsymbol{u},\boldsymbol{v}) \;=\; \frac{\pi\,(-1)^{2\epsilon_u+2\epsilon_v}}{A(\boldsymbol{\beta}_v-\boldsymbol{\beta}_u)A(\boldsymbol{\alpha}_v-\boldsymbol{\alpha}_u)}\,. \tag{5.8}$$

Using this relation one obtains for (5.6)

$$c(\boldsymbol{\rho}_q)^{(N-1)(N-2)/2} \prod_{j=1}^{N-2} \prod_{k=1}^{N-1-j} \varphi(\boldsymbol{u}_k, \boldsymbol{v}_{N-j}) \tag{5.9}$$
$$\times \int dy'\, e^{-iqy'} \int dy\, e^{ipy} \left[\Lambda_2(\boldsymbol{v}_1)^\dagger\, \Lambda_2(\boldsymbol{u}_1)\, \ldots \Lambda_2(\boldsymbol{v}_{N-1})^\dagger \Lambda_2(\boldsymbol{u}_{N-1})\right] (y'|\, y)\,,$$

where we suppose that $\boldsymbol{u}_k \neq \boldsymbol{v}_j$ if $k \neq j$. The calculation of the convolution of the Λ functions in the second line of (5.9) (which can be represented as a chain of the box diagrams) is relied on the identity

$$\int dy\, e^{ipy} \left[\Lambda_2(\boldsymbol{v})^\dagger \Lambda_2(\boldsymbol{u})\right] (x|\, y) \;=\; c(\boldsymbol{\rho}_q) \frac{e^{ipx}}{|p|} (2\pi)^2\, \delta_{\epsilon_u \epsilon_v}\, \delta(u-v)\,. \tag{5.10}$$

Collecting all factors together, one derives for (5.6)

$$\left[c(\boldsymbol{\rho}_q)^{N(N-1)/2} \prod_{m=1}^{N-2} \prod_{j=m+1}^{N-1} \varphi(\boldsymbol{u}_m, \boldsymbol{u}_j)\right] (2\pi)\delta(p-q) \prod_{k=1}^{N-1} \delta(\boldsymbol{u}_k - \boldsymbol{v}_k)\,, \tag{5.11}$$

where $\delta(\boldsymbol{u}-\boldsymbol{v}) = (2\pi)^2 \delta_{\epsilon_u \epsilon_v} \delta(u-v)$. Restoring the symmetry with respect to the permutations of variables $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1}$ ($\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{N-1}$), which is broken due to the imposed condition, $\boldsymbol{u}_k \neq \boldsymbol{v}_j$ for $k \neq j$, one arrives to the expression in the rhs of Eq. (5.4) with

$$\mu_N(\vec{\boldsymbol{u}}) \;=\; \frac{1}{(N-1)!} \left[c(\boldsymbol{\rho}_q)^{-N(N-1)/2} \prod_{m=1}^{N-2} \prod_{j=m+1}^{N-1} \omega(\boldsymbol{u}_m, \boldsymbol{u}_j)\right]\,, \tag{5.12}$$

where

$$\omega(\boldsymbol{u}_m, \boldsymbol{u}_j) = \varphi(\boldsymbol{u}_m, \boldsymbol{u}_j)^{-1} = \frac{1}{\pi} \frac{u_m - u_j}{2} \left[\tanh \pi \left(\frac{u_m - u_j}{2}\right)\right]^{1-4(\epsilon_m + \epsilon_j)}\,. \tag{5.13}$$

We remind here that $(\epsilon_m + \epsilon_j)$ is equal to zero if $\epsilon_m = \epsilon_j$ and to 1/2 otherwise.

The measure $\mu_N(\vec{\boldsymbol{u}})$ is a regular nonnegative function for real $\boldsymbol{u}_k$. When one of the separated variables u_k goes to infinity, it grows as

$$\mu_N(\vec{\boldsymbol{u}}) \overset{u_k \to \pm\infty}{\sim} |u_k|^{N-2} \left(1 + \mathcal{O}(1/|u|)\right)\,. \tag{5.14}$$

Under the analytic continuation to the complex plane, the measure (5.12) becomes a meromorphic function of the variables u_k. One can also verify that it satisfies the functional relation

$$\frac{\mu_N(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k + \mathbf{i}, \ldots, \boldsymbol{u}_{N-1})}{\mu_N(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k, \ldots, \boldsymbol{u}_{N-1})} \;=\; \prod_{j \neq k} \frac{u_k - u_j + i}{u_k - u_j}\,. \tag{5.15}$$

To determine the wave function in the SoV representation, $\Phi_{q_2,\ldots,q_N}(p, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1})$, one calculates the scalar product of the eigenfunction in the coordinate representation, $\Psi_{\mathbf{q}}(\vec{x})$, (Eq. (5.1)) with $U_{p,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N-1}}(\vec{x})$, (Eq. (5.3)). Using Eq. (5.4) one finds

$$\Phi_{\mathbf{q}}(p, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N-1}) \;=\; c_\Psi(p)\, Q_{\mathbf{q}}(\boldsymbol{u}_1)^* \ldots Q_{\mathbf{q}}(\boldsymbol{u}_{N-1})^*\,, \tag{5.16}$$

where

$$c_\Psi(p) \;=\; c_N(p) \int dx_1 \ldots dx_N\, e^{-ipx_1} \Psi_{\mathbf{q}}(x_1, \ldots, x_N)\,. \tag{5.17}$$

Assuming that the basis formed by the functions $U_{p,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N-1}}(\vec{x})$ is complete, one derives the orthogonality condition for the wave functions in the SoV representations

$$\int \prod_{k=1}^{N-1} \mathcal{D}\boldsymbol{u}_k \, \mu_N(\vec{\boldsymbol{u}}) \prod_{k=1}^{N-1} Q_{\mathbf{q}'}(\boldsymbol{u}_k) Q_{\mathbf{q}}(\boldsymbol{u}_k)^* \sim \delta_{\mathbf{q}\mathbf{q}'} \,, \tag{5.18}$$

where we took into account that the integral over p is factorized.

6. Special case: $N = 2$

In the case of the two-sites spin chain, the eigenvalues of the Baxter $\mathbb{Q}-$operator can be obtained in a closed form. Indeed, for the $N = 2$ spin chain, the eigenfunctions are fixed by the group properties, Eqs. (A.4), (A.9). Henceforth, for brevity, we shall restrict ourselves to the case of the representation of the positive parity, i.e. we shall imply that $\epsilon_q = 0$. Applying the Baxter $\mathbb{Q}-$operator to the eigenfunctions and going through the calculations, one finds the representation of the Baxter function in the form of a one-dimensional integral. We shall denote the eigenvalue of the Baxter operator corresponding to the eigenfunction of the continuous series $(\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_q\boldsymbol{\rho}_q}(x, y_1, y_2))^*$, (see Eq. (A.4)) by $Q^{\varepsilon}_{\rho}(u, \epsilon)$ [7]. The eigenvalues of the Baxter operator corresponding to the eigenfunctions of the discrete series are the same, $\mathbb{Q}(u, \epsilon)(\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(w, x_1, x_2))^* = Q_h(u, \epsilon)(\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(w, x_1, x_2))^*$. Up to unessential here $\boldsymbol{u}-$independent factors the eigenvalues of the Baxter operator are given by the following expressions

$$\begin{aligned} Q_h(u, \epsilon) &= \widehat{Q}_h(u, \epsilon) + (-1)^h \, \widehat{Q}_h(-u, \epsilon) \,, &(6.1)\\ Q^{\varepsilon}_{\rho}(u, \epsilon) &= \widehat{Q}^{\varepsilon}_{\rho}(u, \epsilon) + (-1)^{2\varepsilon} \widehat{Q}^{\varepsilon}_{\rho}(-u, \epsilon) \,, &(6.2)\end{aligned}$$

where

$$\begin{aligned} \widehat{Q}_h(u, \epsilon) &= \int_0^1 d\tau \tau^{iu-1} \left[q_h(\tau) + (-1)^{2\epsilon} \, q_h(-\tau)\right] \,, &(6.3)\\ \widehat{Q}^{\varepsilon}_{\rho}(u, \epsilon) &= \int_0^1 d\tau \, \tau^{iu-1} \left[q^{\varepsilon}_{\rho}(\tau) + (-1)^{2\epsilon} q^{\varepsilon}_{\rho}(-\tau)\right] \,. &(6.4)\end{aligned}$$

The functions q_h and q^{ε}_{ρ} can be expressed in terms of the Legendre functions of the second kind as follows

$$q_h(\tau) = \left(\frac{|\tau|}{(1-\tau)^2}\right)^{1-s_q} \mathbf{Q}_{h-1}\left(\frac{1+\tau}{1-\tau}\right) \,, \tag{6.5}$$

$$q^{\varepsilon}_{\rho}(\tau) = \left(\frac{|\tau|}{(1-\tau)^2}\right)^{1-s_q} \left[\lambda(\rho, \varepsilon) \, \mathbf{Q}_{-1/2+i\rho}\left(\frac{1+\tau}{1-\tau}\right) + \lambda(-\rho, \varepsilon) \, \mathbf{Q}_{-1/2-i\rho}\left(\frac{1+\tau}{1-\tau}\right)\right] \,, \tag{6.6}$$

where

$$\lambda(\rho, \varepsilon) = 1 + (-1)^{2\varepsilon} \frac{i}{\sinh \pi\rho} \,. \tag{6.7}$$

[7] Since we assume that $\boldsymbol{\rho_q} = (\rho_q, 0)$, the irreducible representation of the positive parity ($\epsilon = 0$) appears in the tensor product decomposition, therefore $\boldsymbol{\rho} = (\rho, 0)$. We remind also that the parameter $\varepsilon = 0, 1/2$ marks two different eigenfunctions having the same value of the two-particle Casimir operator.

The argument of the Legendre function entering the integrals (6.3), (6.4) varies in the range $[-1, \infty]$. We remind here that for real x, $-1 < x < 1$, the Legendre function is defined by the relation $\mathbf{Q}_\nu(x) = \frac{1}{2}(\mathbf{Q}_\nu(x + i\epsilon) + (\mathbf{Q}_\nu(x - i\epsilon))$.

The factors $(-1)^h$, $(-1)^{2\varepsilon}$ entering Eqs. (6.1), (6.2) determine the parity of the eigenfunctions with respect to the permutation of the arguments. Therefore, one concludes that the eigenvalue of the Baxter operator is even (odd) function of u for the eigenfunction of positive (negative) parity.

Further, it is evident that the function $\widehat{Q}(u)$ ($\widehat{Q}(-u)$) has poles in the upper (lower) half-plane. The poles of $\widehat{Q}(u)$ occur at the points where the integrals (6.3) start to diverge at $\tau \to 0$. Noticing that the integrand in (6.3), (6.4) is an even ($\epsilon = 0$) or odd ($\epsilon = 1/2$) function of τ and taking into account the properties of Legendre functions, one concludes that the function $Q(u, \epsilon)$ has poles of the second order located at the points (4.24), (4.25). It is also straightforward to see that the asymptotic of the Baxter functions (6.1), (6.2) at $u \to \infty$ is in agreement with Eqs. (4.33), (4.34). Calculating the difference $Q(u, 0) - Q(u, 1/2)$ one finds that it can be represented as

$$\sim \int_0^\infty d\tau \tau^{iu-1} q(-\tau)\,, \tag{6.8}$$

where $q(\tau)$ is given by (6.5) or (6.6). Since the function $q(\tau)$ is analytic near $\tau = -1$ we reproduce (4.35).

At last, to find the energies of the corresponding eigenstate it is necessary to calculate the expansions of the Baxter functions near the first poles, $u = \pm i(1 - s_q)$ (see. Eq. (4.22)). Taking into account that for $\tau \to 0$

$$\mathbf{Q}_\nu \left(\frac{1+\tau}{1-\tau} \right) = -\frac{1}{2} \log |\tau| + \psi(1) - \psi(1+\nu) + \mathcal{O}(\tau) \tag{6.9}$$

one finds, e.g. for $\widehat{Q}^\varepsilon_\rho(-i(1 - s_q) + u, 0)$,

$$\widehat{Q}^\varepsilon_\rho(-i(1 - s_q) + u, 0) = 2 \int_0^1 \frac{d\tau}{\tau^{1-iu}} \left[-\log \tau + \right. \tag{6.10}$$
$$\left. \lambda(\rho, \varepsilon)[\psi(1) - \psi(1/2 + i\rho)] + \lambda(-\rho, \varepsilon)[\psi(1) - \psi(1/2 - i\rho)] \right] + \mathcal{O}(u)\,.$$

Then using Eq. (4.22) and taking into account that $\mathcal{H}_2 = 2H_{12}$, one reproduces the expressions (3.29).

7. Summary

In this paper we studied the spin chain model with the $SL(2, \mathbb{R})$ symmetry group. The Hilbert space attached to each site is $L^2(\mathbb{R})$ and the symmetry transformations are realized by the operators of the unitary principal series continuous representation of the $SL(2, \mathbb{R})$ group. To define the model we constructed the $SL(2, \mathbb{R})$ invariant solution of the Yang-Baxter equation – the $\mathcal{R}-$operator which acts on the tensor product of two $L^2(\mathbb{R})$ spaces. The Hamiltonian of the model is defined as the derivative of the fundamental transfer matrix and is given by the sum of the two-particle Hamiltonians. The pair-wise Hamiltonians have both the discrete and continuous spectrum, Eq. (3.29), which reflects the pattern of the decomposition of the tensor

product of two representations of the principal continuous series into irreducible components. The eigenstates belonging to the continuous spectrum are specified by the value of the $sl(2)$ spin, $s = 1/2 + i\rho$, and parity with respect to the permutation of the coordinates x_1 and x_2. The energy gap between the eigenstates with the same value of the spin and different parity is maximal for $\rho = 0$ and exponentially decreases with ρ.

To solve the model we applied the method of the Baxter $\mathbb{Q}-$operator [2] and the Separation of Variables [4]. The standard ABA approach [1] is not applicable for the model in question due to the absence of the lowest weight vector in the Hilbert space of the model. Having realized the Baxter operator as an integral operator we resolved the defining equations and obtained the kernel in the explicit form. It allowed to determine the properties of the eigenvalues of the Baxter operator as functions of the spectral parameter. Then the eigenvalues of the Baxter operator can be obtained as the solutions of the Baxter equation in the certain class of functions. The eigenvalue of the Baxter operator encodes all information about the corresponding eigenstate. We have shown that the Hamiltonian of the model can be obtained as a derivative of the Baxter operator at special points. Moreover, the arbitrary transfer matrix factorizes into the product of two Baxter $\mathbb{Q}$ operators at certain values of the spectral parameters. Analoguous results have been obtained for the noncompact $SL(2,\mathbb{R})$ (discrete series) and $SL(2,\mathbb{C})$ (continuous series) spin chains, see Refs. [7, 11].

We have constructed the representation of the separated variables for the model in question. The kernel of the unitary operator, which maps the eigenfunction to the SoV representation, has been obtained in an explicit form. It factorizes into the product of $N-1$ (N is the number of sites) operators each depending on one separated variable only. The kernel of the transition operator can be visualized as a Feynman diagram with a specific pyramidal form. This form of the kernel, first obtained for the $SL(2,\mathbb{C})$ spin chain [11], is a general feature of all noncompact $SL(2)$ spin magnets [12, 15]. Using the diagram technique we calculated the scalar product of the transition kernels and determined the Sklyanin's integration measure. We have shown that the wavefunction in the separated variables is given by the product of the eigenvalues of the (conjugated) Baxter operator. Therefore the knowledge of the eigenvalue of the Baxter operator allows to restore the eigenfunction.

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A Appendix: Tensor product decomposition

In this Appendix we collect the necessary formulae concerning the decomposition of the tensor product of the representations of the principal continuous series. The tensor product of two representations of the principal continuous series is decomposed into the direct integral over the representations of the same type and into the direct sum over the unitary representations of the discrete series [26]. We remind that the representations of the discrete series $\mathcal{D}_h^{\pm}$ are labelled by

the (half)integer number, h, and can be realized by the unitary operators $D_h^+(g)$ ($D_h^-(g)$)

$$\left[D_h^{\pm}(g)\Psi_{\pm}\right](w) \;=\; (cw+d)^{-2h}\,\Psi_{\pm}\left(\frac{aw+b}{cw+d}\right), \tag{A.1}$$

acting on the Hilbert space $\mathcal{H}_h^{\pm}$, respectively. The latter are defined as the space of the functions analytic in upper ($\mathcal{H}_h^+$), lower ($\mathcal{H}_h^-$) half-plane with the scalar product defined by the following expression [25]

$$\langle \Psi_{\pm}|\Phi_{\pm}\rangle \;=\; \int \mathcal{D}_{\pm}w\,\overline{\Psi_{\pm}(w)}\,\Phi_{\pm}(w)\,, \tag{A.2}$$

where

$$\mathcal{D}_{\pm}w = \frac{2h-1}{\pi}\theta(\pm\operatorname{Im}(w))(\pm\operatorname{Im}(w))^{2h-2}dxdy \tag{A.3}$$

and $w = x+iy$.

The operators separating the irreducible components in the tensor product $T^{\boldsymbol{\rho}_1}\otimes T^{\boldsymbol{\rho}_2}$ are the following: the projectors to the discrete series, $\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}$, which map $T^{\boldsymbol{\rho}_1}\otimes T^{\boldsymbol{\rho}_2}\to D_h^{\pm}$, and the projectors $\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}$ to the principal continuous series. The projecting operators can be realized as integral operators. The irreducible representation $T^{\boldsymbol{\rho}}$, ($\boldsymbol{\rho} = (\rho,\epsilon)$, $\rho>0$, $\epsilon=\epsilon_1+\epsilon_2$) enters into the tensor product $T^{\boldsymbol{\rho}_1}\otimes T^{\boldsymbol{\rho}_2}$ with double multiplicity. Therefore one can construct two projectors, $\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}$, where the parameter ε takes two values 0 and 1/2 and marks these equivalent representations. The integral kernel of the projector is given by the product of three propagators

$$\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(x,y_1,y_2) \;=\; D_{\boldsymbol{\alpha}_3}(y_2-x)D_{\boldsymbol{\alpha}_2}(x-y_1)D_{\boldsymbol{\alpha}_1}(y_1-y_2)\,, \tag{A.4}$$

where the indices are defined as follows

$$\begin{aligned}
\boldsymbol{\alpha_1} &= (\frac{1}{2}-i(\rho+\rho_1+\rho_2),\varepsilon+\epsilon_1),\\
\boldsymbol{\alpha_2} &= (\frac{1}{2}+i(\rho+\rho_2-\rho_1),\varepsilon),\\
\boldsymbol{\alpha_3} &= (\frac{1}{2}+i(\rho+\rho_1-\rho_2),\varepsilon+\epsilon_1+\epsilon_2)\,.
\end{aligned} \tag{A.5}$$

Then the function $\Phi^{\boldsymbol{\rho},\varepsilon}$ defined as

$$\Phi^{\boldsymbol{\rho},\varepsilon}(x) \;=\; \int dy_1dy_2\,\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(x,y_1,y_2)\,\Psi(y_1,y_2), \tag{A.6}$$

transforms according to the representation $T^{\boldsymbol{\rho}}$. Next, using the techniques for the calculation of the $SL(2,\mathbb{R})$-integrals introduced in Sec. 3., it is straightforward to check the following orthogonality condition

$$\int dy_1\,dy_2\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(x,y_1,y_2)\left(\Pi^{\boldsymbol{\rho}',\varepsilon'}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(x',y_1,y_2)\right)^* \;=\; \mathcal{N}(\boldsymbol{\rho})\,\delta_{\varepsilon\varepsilon'}\,\delta(x-x')\,\delta(\rho-\rho'), \tag{A.7}$$

where

$$\mathcal{N}(\boldsymbol{\rho}) \;=\; (2\pi)^2\rho^{-1}\coth^{1-4\epsilon}(\pi\rho)\,. \tag{A.8}$$

The conformal spin h of the representations of the discrete series $D_h^{\pm}$ entering the decomposition of the tensor product $T^{\boldsymbol{\rho}_1} \otimes T^{\boldsymbol{\rho}_2}$ is integer for $\epsilon_1 = \epsilon_2$, and half-integer otherwise. The projecting operators onto irreducible components are

$$\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(w,x_1,x_2) \;=\; \frac{(x_2-x_1)^{h+\epsilon_1+\epsilon_2}}{(x_1-w)^{h+\epsilon_2-\epsilon_1}(x_2-w)^{h+\epsilon_1-\epsilon_2}}\;|x_1-x_2|^{i(\rho_1+\rho_2)-1-\epsilon_1-\epsilon_2} \tag{A.9}$$
$$\times\; \left[\frac{x_1-w}{x_2-w}(x_1-x_2)\right]^{[i(\rho_1-\rho_2)-\epsilon_1+\epsilon_2]/2} \left[\frac{x_2-w}{x_1-w}(x_2-x_1)\right]^{[i(\rho_2-\rho_1)-\epsilon_2+\epsilon_1]/2},$$

where the w lies in the upper half-plane for the $''+''$ projector, and in the lower for $''-''$ projectors. Note that the expressions in the square brackets in (A.9) are the single–valued functions of w in the upper or lower half-plane. Thus the functions $\Phi^{h,\pm}$

$$\Phi^{h,\pm}(x) \;=\; \int dy_1 dy_2\, \Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(w,y_1,y_2)\,\Psi(y_1,y_2) \tag{A.10}$$

transform according to the representations $\mathcal{D}_h^{\pm}$. The normalization condition reads

$$\int dx_1 dx_2 \left(\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(w,y_1,y_2)\right)^* \Pi^{h',\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(z,y_1,y_2) \;=\; c(h)\,\delta_{hh'}\,\mathbb{K}(z,w), \tag{A.11}$$

where the $\mathbb{K}(z,w) = e^{i\pi h}(z-\bar{w})^{-2h}$ is the reproducing kernel (unit operator on $\mathcal{H}_h^{\pm}$) and

$$c(h) = (2\pi)^2 \frac{\Gamma(2h-1)}{\Gamma^2(h)}\,. \tag{A.12}$$

Then any function $\Psi(x_1,x_2) \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ can be decomposed as

$$\Psi(x_1,x_2) \;=\; \sum_{h=1+(\epsilon_1+\epsilon_2)/2}^{\infty} c^{-1}(h) \int \mathcal{D}_{\pm} w \left(\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(w,y_1,y_2)\right)^* \Phi^{h,\pm}(w) \;+$$
$$\sum_{\varepsilon=0,1/2} \int_0^{\infty} d\rho\, \mathcal{N}^{-1}(\boldsymbol{\rho}) \int_{-\infty}^{\infty} dx \,\left(\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_2}(x,x_1,x_2)\right)^* \Phi^{\boldsymbol{\rho},\varepsilon}(x)\,, \tag{A.13}$$

where the functions $\Phi^{h,\pm}$ and $\Phi^{\boldsymbol{\rho},\varepsilon}$ are defined in Eqs. (A.10), (A.6), respectively.

Note, that if $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2$, the projectors have definite parity with respect to the permutation of the coordinates x_1 and x_2,

$$\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_1}(x,x_1,x_2) \;=\; (-1)^{2\varepsilon+2\epsilon_1}\,\Pi^{\boldsymbol{\rho},\varepsilon}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_1}(x,x_2,x_1)$$

and

$$\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_1}(w,x_1,x_2) \;=\; (-1)^{h+2\epsilon_1}\,\Pi^{h,\pm}_{\boldsymbol{\rho}_1\boldsymbol{\rho}_1}(w,x_2,x_1)\,.$$

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